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# A Study of the Interaction of Insurance and Financial Markets: Efficiency and Full Insurance Coverage

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**Abstract**

The first contribution of this article is to provide a framework, a model together with a corresponding equilibrium notion, suitable for the study of the interaction between insurance and dynamic financial markets. Our central result is that in equilibrium risk-averse agents purchase full insurance coverage, despite unfair insurance prices. We identify three conditions that explain this result: (1) insurance contracts are priced competitively, (2) financial prices include a risk premium only for undiversifiable risk, and (3) financial markets are effectively complete. An implication is that in this model disasters can be insured by fully assessable stock insurance companies.

*Consumers trading in both markets at once use the financial market to diversify their investment portfolio and use the insurance market to insure their personal risk. In ignoring trade in financial assets, the formal model in this article bypasses an important aspect of consumer behavior under risk.*

Marshall (1974b, p. 675)

## Introduction

This article provides a novel framework with which to study insurance without "bypassing trading in financial markets." Including financial markets as an integral part of the analysis provides rich insights on the workings of modern insurance markets. The first such insight, and the main contribution of this article, is that when financial markets are sufficiently flexible, efficiency, and equilibrium go hand in hand with

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full insurance coverage even in the presence of loaded insurance prices. This article shows that it is optimal to purchase full coverage, even when insurance prices are not actuarially fair. This is demonstrated using a general equilibrium model, where agents not only buy insurance but can also invest in shares of the companies that sell insurance. Moreover, it is shown that equilibrium is efficient in the sense that it decentralizes a Pareto optimal risk sharing rule.

To illustrate the main ideas in this article, consider the following simple example:[1] there are 50 villagers living on an island that is the only source of coconuts for the nearby archipelago. Each villager owns a coconut tree that will produce 1,000 coconuts in one year's time. All villagers are exposed to the risk of losing 80 percent of the coconuts if their tree is hit by lightning, which occurs with 10 percent probability per year (assume the risks are independent from one tree to another). Suppose that insurance is issued by three insurance companies that compete fiercely for customers in prices. Insurance is sold at the beginning of the year, it is not traded during the year, and the corresponding indemnities are paid at the end of the year. Insurance companies also issue shares in the island's booming stock market, which is open every day, and where a riskless bond is also traded. Anyone holding shares of an insurance company at the end of the year will receive a dividend equal to the insurance premia collected by the insurance company at the beginning of the year minus the indemnity payments that are due—if indemnity payments are greater than the premia collected, the shareholders will have to make up the difference.

The results in this article say that in an equilibrium of this economy all villagers will buy full coverage (i.e., receive 800 coconuts in the event of a loss) and they will pay a premium that will be more than 80 coconuts, say 90 coconuts, to the insurance companies. The 10 coconuts the insurance companies receive from each villager over and above the actuarially fair price of 80 will be redistributed to the insurance companies' shareholders to compensate them for the risk of owning shares—the risk that indemnity payments are not equal to their expected value and that the indemnities tend to be high (and dividends low) when coconuts are scarce. This article shows that all the villagers will own shares and their asset trading behavior can be described using a single fully diversified portfolio, for example, a portfolio that puts the same weight on each of the three companies. But, if villagers have different degrees of risk aversion, they will invest different amounts of wealth in the portfolio relative to the riskless bond: those with lower aversion to risk will have more wealth invested in risky insurance company stocks. As time passes and storms come and go, sometimes lightning will hit some trees and sometimes not. These events affect the probability and amount of indemnities each insurance company will pay and hence the dividends to shareholders and current share prices. Villagers will react to these changes in share prices and future dividends by readjusting the amount of money invested in the risky portfolio. The current article shows that not only is the villager's behavior just described

[1] All numbers for the example are made up (coconut trees only produce around 50 coconuts per year) but they are representative of the kind of numbers that can be obtained from a numerical example. The interest rate is assumed to be zero for convenience.

optimal but also that the economy's institutions, private insurance combined with trading on insurance company shares, leads to an optimal distribution of the risk in the island amongst all villagers.

Our analysis starts by setting up a general model of an economy in which private insurance is available and insurance company shares are traded. Then, we define an appropriate notion of equilibrium. Having developed the necessary tools, we present the main results. First, it is shown that there exist efficient insurance market equilibria and that the number of actively traded financial assets needed to attain efficient allocations can be as few as two. This is followed by the central result of the article: in efficient equilibria agents buy full insurance. Our main result is shown to hold despite strictly positive loadings on insurance. The article identifies three conditions that together explain why buying full insurance is optimal for any risk-averse individual even in the presence of loaded insurance prices. These three conditions are formalized in the article and can be described as requiring that (i) insurance contracts be priced competitively, (ii) financial prices include a risk premium only for undiversifiable risk, and (iii) financial markets are effectively complete. Conditions (i) and (ii) ensure that the price of insurance is "economically fair"; i.e., it exactly compensates for the actuarial and the economic risk (the probability and magnitude of indemnity payments plus the market price for the undiversifiable component of insurance risk, minus the time value of premium payments), and Condition (iii) says that given a sufficient amount of initial capital, any Pareto efficient consumption allocation can be constructed by trading in financial markets. These results imply that in this model disasters can be insured by stock insurance companies when those companies are fully assessable (owners are liable for any liabilities not covered by firm assets, as are private names for the liabilities of their syndicates at Lloyd's of London).

Another result obtained here is to show that optimal trading strategies, which are usually obtained as the solution of a stochastic differential equation, can be described explicitly in terms of (deterministic) equations.

Methodologically, the article makes two contributions: the model and the notion of equilibrium. The model developed in this article generalizes those in the insurance literature by allowing the joint analysis of the market for private insurance and continuously open financial markets. The model also permits great heterogeneity in preferences and endowments. The insurance market equilibrium is an equilibrium concept especially suited for problems of insurance, where contracts that are not traded dynamically (personal insurance) coexist with actively traded financial ones (bonds and stock company shares).

A final contribution of this article is to provide a unifying framework for a number of existing results. For example, as markets are frictionless and there are no agency costs, reinsurance is redundant as put forward in Doherty and Tiniç (1981). In fact, as investors will always hold a fully diversified portfolio of shares, the number of insurance companies is irrelevant (as long as they act competitively). A second example is the characterization of the loading using the market price for risk and its actuarial properties. This is the approach pursued in Ellickson and Penalva (1997), Aase (1999), and Schweitzer (2001), and it contrasts with the approach that the loading is determined from the insured's willingness to pay or the interaction of the risk aversions of insurance companies and reinsurers (as in many articles from Borch, 1962, to Aase, 2002).[2]

The article proceeds as follows. This introduction concludes with a brief overview of some of the more relevant articles in the literature. The next section introduces the model. As the model is new, the section provides a detailed description of its different aspects: preferences and risks, the insurance market and the stock market (contracts and pricing), and agents' budget constraints. The section concludes with a short review of the model plus the definition of an insurance market equilibrium. Then, the first results on efficiency are presented in the second section. These set the stage for the central result in the third section, where the characteristics of efficient equilibria are determined, and it is shown that it is optimal to buy full insurance coverage despite unfair prices. The fourth section analyzes the causes behind the central result and identifies the three conditions that help explain the insurance decision, as well as including other interesting results. The final section puts the efficiency of equilibrium result in context, discusses the special case where agents have HARA preferences, and concludes with some comments on future research.

Only the short and simpler proofs are in the text, the rest are in the Appendix.

## RELATED LITERATURE

The central result in this article formalizes for the first time the following description of equilibrium in Kihlstrom and Pauly (1971): "Persons who bear part of the total loss might be thought of as having a kind of split personality in which they make a certain payment in return for coverage which does not depend on total loss but in which they hold 'stock' in an insurance 'firm' which makes their final wealth positions vary with the total loss" (quotes in the original). In the current article insurance firms are explicitly included and the results in Kihlstrom and Pauly (1971) are greatly generalized. Also, the analysis goes further by identifying the conditions under which equilibrium implies full insurance demand and show that these conditions could also hold out of equilibrium.

While the main contribution of the current article is entirely original, it also contains additional results that extend and complement a large number of results in the existing literature. We will restrict attention to the closest references. For example, there is an extensive literature on the efficiency of insurance markets.[3] Borch (1962) and Wilson (1968) established the mutuality principle for characterizing efficiency under risk. Marshall (1974a) discusses how this (mutuality) approach to insurance provides a solution to the provision of catastrophic insurance that cannot be obtained from a reserves-based approach (which relies on the Law of Large Numbers)—an argument

[2] MacMinn and Witt (1987) analyze the determination of the loading with monopolistically competitive and risk-averse insurance firms that invest in a competitive financial market. These firms set premium prices that will depend on the aggregate price of risk and their corporate risk aversion and the loading they obtain equals the one obtained here for the special case of risk-neutral insurers.

[3] The relationship between the efficiency results in this article and existing ones is analyzed in detail in the "Insurance Equilibrium and Effective Completeness" section.

repeated in different forms in the literature and one that motivates our analysis of the role of financial markets as an institution to implement efficient risk sharing. Based on these early results, many researchers have studied mechanisms other than financial markets to implement efficient risk sharing arrangements, e.g., via mutual insurance companies (Doherty and Dionne, 1993), via the design of private insurance contracts (Cass Chichilninsky and Wu, 1996; Cummins and Mahul, 2003), or via reinsurance (Doherty and Tiniç, 1981; Borch, 1984; Jaffee and Russell, 1997; Froot, 2001; Zanjani, 2002).

The financial market based approach to insurance is studied in Ellickson and Penalva (1997), Harrington and Niehaus (1999), Aase (2001), Christensen, Graversen, and Miltersen (2001), and Penalva (2001). The current article extends the literature by formally including insurance as a nontradable asset and studying its price and demand. The interaction of individual insurance and investment decisions is treated in Smith and Mayers (1983), Eeckhoudt, Meyer, and Ormiston (1997), and Somerville (2004), as a partial equilibrium problem while Penalva takes a general equilibrium perspective. The present analysis is a general equilibrium one and differs from Penalva in a number of ways. Most importantly, in this article insurance contracts are not tradable. Also, Penalva looks for conditions that will ensure efficiency while the current article focuses on the details of insurance demand, investment decisions, and prices. Technically, this article also differs in that the proofs used here are more elegant and constructive.

## The Framework

This section contains the basic methodological contribution of the article: a model of private risks with nontraded private insurance and a financial market with continuous trading, plus the notion of a competitive insurance market equilibrium. The model and the equilibrium notion provide the basic tools with which the economic results of the article are demonstrated.

The following two subsections describe in detail all the different components that define the model. The eager reader may jump ahead to Section "Model Overview and Equilibrium," where the model is summarized briefly and the notion of a competitive insurance market equilibrium is defined, and refer back to the sections "The Basic Economy: Preferences and Risk" and "Insurance and the Stock Market" as needed.

### The Basic Economy: Preferences and Risk

The basic model is that of a two-date economy, $\mathcal{E}$, with a finite number of agents who have heterogeneous von Neumann–Morgenstern preferences and risky endowments (extensions are considered in the section "HARA Preferences and Linear risk Sharing Rules").

*Preferences.* In this economy there are $n < \infty$ agents with heterogenous preferences described by standard risk-averse expected utility functions:

**Assumption 1:** *For all $i = 1, \ldots, n$,*

$$U_i(x) = u_i(x(0)) + \beta_i E[v_i(x(1))],$$

*where both $u_i$ and $v_i$ are increasing, strictly concave differentiable functions satisfying the standard Inada conditions (described in the Appendix).*

*Endowments.* Each agent, $i$, has a constant income at each date ($w_{i,0}$, $w_{i,1}$). At some point between dates zero and one, each agent may suffer an accident (at most) that translates into a fixed loss, $L$, of date one income. The probability that agent $i$ has an accident between dates 0 and 1 is equal to $p$.

*Risk.* The random variable $N_i(t)$ keeps track of the number of accidents suffered by agent $i$ between date 0 and $t \in [0, 1]$. As each agent can have at most one accident, $N_i(t)$ can only take values 0 or 1. The vector $\mathbf{N}(t) \equiv (N_1(t), \ldots, N_n(t))$ keeps track of each agent's accidents, and $N(t) \equiv \sum_{i=1}^{n} N_i(t)$ keeps track of the total number of accidents.

Let $e_i \equiv (e_i(t))_{t\in\{0,1\}}$ denote agent $i$'s endowment and $e \equiv (e(t))_{t\in\{0,1\}}, e(t) = \sum_{i=1}^{n} e_i(t)$, the aggregate endowment. Then:

**Assumption 2:** *Agent endowments are:*

$$e_i(0) = w_{i,0} \quad \& \quad e_i(1) = w_{i,1} - N_i(1)L,$$

*where $Pr(\{N_i(1) = 1\}) = p \in (0, 1)$, $L > 0$, and $w_{i,0} \geq 0$, $w_{i,1} \geq L$ with strict inequality for at least one $i$. All agents have the same priors on the distribution of $\mathbf{N}$.*

## Insurance and the Stock Market

In addition to preferences and endowments, the model includes two sectors: a sector with nontraded contracts (private insurance) and a sector with continuously trading contracts (financial market).

*Nontraded Contracts.* A basic characteristic of private insurance is that it is provided in the form of private, personalized, nontransferable contracts. Insurance contracts are contracts settled via bilateral negotiation between an individual and a corporation (or its representatives) that specifies payments to the individual depending on certain individual-specific occurrences that cannot be transferred to a third party.[4]

Let $\mathcal{I}$ denote the set of nontraded contracts, which is the union of the set of nontraded contracts available to each individual, $\mathcal{I}_i$. Each agent can purchase private insurance, a contract that pays him only in the event that he has an accident (the event: $\{N_i(1) = 1\}$) and can choose what amount he wants reimbursed, which can be anything between nothing (no coverage) and all of $L$ (full coverage).[5]

[4] Insurance markets are extremely sophisticated. The description just provided applies quite generally but is highly simplified. A general definition of an insurance contract would include caveats to account for the current richness and complexity of the practice of insurance contracts: contracts who cover more than one individual, whose payments depend on special nonindividual specific events, contracts with clauses that allow the transfer of the contract to certain prespecified third parties, etc.

[5] Thus, for all $i, \mathcal{I}_i = \{(\alpha N_i(1))_{\alpha\in[0,L]}\}$.

*Insurance Companies and Traded Contracts.* An insurance company sells private insurance contracts and can issue financial contracts that are tradable in stock markets (insurance company shares, risk debt, etc.). Financial markets are frictionless and all claims issued by the insurance company will be satisfied; i.e., there is no bankruptcy because investors in insurance companies are fully assessible for the obligations of the companies. In such a setting, the classic result of Modigliani and Miller (1958) applies: the value of the insurance firm is independent of its financing strategy. For simplicity we assume there are $J$ insurance companies who are fully equity financed and each company issues (infinitely divisible) shares at date zero. The total number of shares is normalized to one. A share is a claim on the premia collected by the insurance company (plus any interest earned between dates 0 and 1) minus any indemnities paid at date 1. It is convenient to assume that all premia are invested in riskless bonds.[6] The claims that will be received by the owner(s) of the shares issued by insurance company $j = 1, \ldots, J$ is represented by a random variable, $d_j$. The price of this claim is determined in financial markets.

*Pricing in Financial Markets.* The shares issued by insurance companies can be traded in a frictionless financial market at any time ($t \in [0, 1]$), called a stock exchange. We allow continuous trading because we want to study the interaction of insurance and investment decisions in a context where investments are made in informationally efficient markets and the market for aggregate risk is complete. Our financial market is an institution in which an auctioneer continuously sets prices to facilitate share trading. We assume there is no private information or agency costs and the auctioneer sets prices such that no arbitrage opportunities exist. Agents can go to the stock market and trade shares at the announced prices at any time without any costs, frictions, or constraints. Agents have common priors (described by an objective probability measure, $P$), so that trades will be motivated purely by the desire to control risk exposures.

No arbitrage in frictionless financial markets implies that there exists a probability measure, $Q$, the risk-neutral probability measure, and an interest rate process, $r(t)$, such that the market value of an asset is its expected value discounted appropriately (Harrison and Kreps, 1979). We use the following notation: for any random variable, $x$, $E_{Q_t}[x]$ is the expectation of $x$ conditional on information available at date $t$ using probability measure $Q$. The relation between the true probabilities $P$ and the risk-neutral probabilities $Q$ is expressed by the Radon-Nikodym derivative $\xi$.[7] The price of a share in company $j$ at date $t$, $S_j(t)$, which promises claims $d_j$ at date 1 is given by:

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[6] In the current frictionless model, this assumption is without loss of generality as the investment strategy pursued by the insurance company with the premia collected will not add or subtract value to the company.

[7] Formally, let $\xi(1) = dQ/dP$ denote the density of $Q$ with respect to $P$, such that for every random variable $x$, $E_Q[x] = E_P[x\xi(1)]$. Also, for $t \in [0, 1)$,

$$\xi(t) = \frac{\xi(1)}{E_{P_t}[\xi(1)]} \equiv \frac{\mathrm{d}Q_t}{\mathrm{d}P_t}.$$

$$\forall t \in [0,1], \quad S_j(t) = E_{Q_t}\left[d_j e^{-\int_t^1 r(s)\,\mathrm{d}s}\right] \quad P\text{-a.s.} \tag{1}$$

Let $\mathbf{r} = \int_0^1 r(t)\,\mathrm{d}t$ be the interest rate, so that the price of the riskless bond that, it is assumed, promises to pay one unit of consumption at date 1, $d_0 = 1$, has price $S_0(t)$ where

$$S_0(t) = e^{-\mathbf{r}(1-t)} \equiv \exp\left(-\int_t^1 r(s)\,\mathrm{d}s\right).$$

The set of assets traded in stock markets (insurance company shares plus the riskless bond) is described by its no-arbitrage prices and denoted by $\mathcal{D} = \{S_j(t)_{t\in[0,1]} \mid j = 0, 1, \ldots, J\}$.

*Information.* Insurance company shares are claims on the performance of those companies, which depends solely on the indemnities they have to pay out. These indemnities themselves depend on who has had accidents. The process of the arrival of accidents to agent $i$ is described by a common hazard rate, $\lambda(t)$, as follows: for any random process $\{x(s)\}_{s\in[0,1]}$ and any $t \in (0, 1]$, define $x(t^-) = \lim_{s\uparrow t} x(s)$.

**Assumption 3:** *For each i, the dynamics of $N_i(t)$, $t \in [0, 1]$ is described by the hazard rate, $\lambda_i(t)$, which is defined by the function $\eta : [0, 1] \times \{0, 1, \ldots, n\} \to \mathbf{R}$:*

$$\lambda_i(t) = \begin{cases} \eta(t, N(t^-)) & \text{if } N_i(t^-) = 0 \\ 0 & \text{if } N_i(t^-) = 1 \end{cases}$$

*To better understand this assumption consider the following three examples:*

**Example 1** (constant hazard): If the hazard rate is constant, $\eta(t, N(t^-)) = \lambda$, then for any $i$ the distribution of the arrival time of an accident to $i$ is i.i.d. exponential with parameter $\lambda$ and the total number of accidents, $N(t)$, is similar to a Poisson process. Note that the parameter $\lambda$ is related to $p$ via the following equation:

$$p = 1 - \exp(-\lambda) \Leftrightarrow \lambda = -\ln(1 - p).$$

**Example 2** (hurricanes): Consider the risk of a hurricane destroying the coconut trees in the example used in the introduction. Describe the risk of a hurricane affecting any one tree using two constants, $\lambda > 0$ and $\gamma > 1$, as follows. At the beginning, if $N(t) = 0$, the hazard rate for every tree is $\lambda$. If one agent's tree is hit by a hurricane at time $s$, so that $N(t) \geq 1$ for $t > s$, then the proximity of a hurricane increases everyone else's hazard rate to $\lambda\gamma$. The hazard rate will then be: $\eta(t, 0) = \lambda$ and $\eta(t, m) = \lambda\gamma$ for $m = 1, \ldots, n$.

**Example 3** (aging): Consider the risk of coconut trees becoming unproductive and assume all trees are planted at the same time. As time passes a tree may randomly become unproductive. This risk is independent from one tree to another and increases with age. One way to incorporate this risk into the model is by assuming that the probability that a tree becomes unproductive between today and tomorrow is increasing with the time since the tree was planted, e.g., let $\eta(t, N(t-)) = \beta t^{\alpha}$ with $\beta > 0$ and $\alpha > 1$.

*Diversified Portfolios.* A special construct that will useful throughout the article is that of a fully diversified portfolio of risky shares. Given a set of prices described by the pair $(Q, r)$, a self-financing portfolio of risky shares, $((\theta_j(t))_{j=1}^{J})_{t\in[0,1]}$ is *fully diversified* if there exists a deterministic one-to-one function $g : \mathbf{R} \to \mathbf{R}$ such that $\sum_{j=1}^{J} \theta_j(1)d_j(1) = g(e(1))$.[8] Then, the portfolio is fully diversified in the sense that dividends from the portfolio only depend on what happens to the economy on aggregate. This property is very important—it is used, among other things, to characterize efficient risk-sharing rules by the mutuality principle (Borch, 1962; Wilson, 1968)— and we will refer to it as *the mutuality property*.

This article will make use of a particular type of fully diversified portfolio: the equally weighted portfolio, a portfolio that puts the same weight in each of the $J$ risky assets (insurance companies), i.e., $\theta_j(t) = 1/J$, $j = 1, \ldots, J$.

*Insurance Pricing.* A key element of the model proposed here is the presence of non-traded contracts, $\mathcal{I}$. Given that we are interested in insurance contracts that are not traded and are negotiated bilaterally it is not obvious how their prices will be determined.

The sustitutability between consumers and between insurance companies suggests that insurance contracts should be priced "competitively." The buyer (insured) is one of $n$ potential insureds, all with the same risk $(p, L)$. The seller (insurance company) is one of $J$ companies, all of which have the same "technology" for providing insurance, at least for sufficiently large $n$ and $J$. Therefore, any of the parties, buyer or seller, can be easily substituted and prices should reflect the lack of bargaining power between the parties. Define competitive insurance prices as follows:

**Definition 1:** *In an economy where the pair $(Q, r)$ prices all traded assets, $\mathcal{D}$, according to Equation (1), insurance prices are competitive under $(Q, r)$ if for all $i$,*

$$S_i^I = e^{-\mathbf{r}} E_Q[N_i(1)]. \tag{2}$$

[8] A portfolio of risky shares is self-financing if

$$\sum_{j=1}^{J} \theta_j(t)S_j(t) = \sum_{j=1}^{J} \theta_j(0)S_j(0) + \sum_{j=1}^{J} \int_0^t \theta_j(s)\,\mathrm{d}S_j(s), \quad \forall t \in [0, 1].$$

Further, we assume that a self-financing fully diversified portfolio satisfies an additional technical condition. Let $S_j(t, N(t) + 1)$ be the price of security $j$, calculated using Condition 1 but changing the current history of events by adding an extra accident (to any of the consumers that have not had an accident yet) at date $t$. Then, a self-financing fully diversified portfolio will be assumed to satisfy the condition: $\sum_{j=1}^{J} \theta_j(t)(S_j(t) - S_j(t, N(t) + 1)) \neq 0$ P-a.s. on $\{N(t) \neq n\}$ for all $t$. This assumption ensures that price of the portfolio always responds to new accidents. This technical condition is generally satisfied by almost all fully diversified portfolios. In fact, a fully diversified portfolio that does not satisfy this additional condition can be made to satisfy it by a very small change in the portfolio weights.

This condition states, literally, that the competitive price of insurance should be determined as if it were a tradable financial asset satisfying the no-arbitrage condition. This condition is not defensible using arguments based on arbitrage (as insurance contracts are highly illiquid assets). But, this condition is justified because it arises as the result of competition (e.g., Bertrand-type successive price cutting) among insurance firms. For example, consider the following thought experiment: a company sells $k$ units of coverage at price at $S_i^I$ to agent $i$. This contract represents a liability: $k \cdot N_i(1)$ of date 1 consumption, and an asset: $kS_i^I e^{\mathbf{r}}$ of date 1 consumption. The effect of selling this contract at date 0 on the price of the insurance company is to change it by $\epsilon \equiv kS_i^I - e^{-\mathbf{r}} E_Q[kN_i(1)]$. This effect cannot be negative as the insurance company can always reject (not propose) such a contract. But, if it is strictly positive; i.e., if $\epsilon > 0$, then another insurance company can offer the same contract at a price of $S_i^I - \epsilon/2k$ and increase its share price by at least $\epsilon/2$. The private insurance market is said to be competitive if the possibility to increase a company's share price using such a strategy does not exist. This then implies that the competitive price of insurance contracts will be equal to $e^{-\mathbf{r}} E_Q[N_i(1)]$ as claimed.

*Insurance and Investment Decisions.* An agent in this economy is faced with three basic decisions: how much private insurance to buy, how and how much to invest in financial assets (bond and insurance companies' shares), and how much to consume. Agent's decisions are restricted in the sense that they cannot spend more than they are endowed with, they have access to a restricted set of insurance contracts (they have to be in $\mathcal{I}_i$) and any investment strategy has to be implementable given prices, his wealth, and his insurance decision.

To reduce notation, for each agent $i$, let $B_i(\mathcal{D}, S_i^I)$ denote agent $i$'s budget constraint, i.e., the set of consumption allocations the agent can achieve given his endowment, prices, and trading opportunities.[9]

## Model Overview and Equilibrium

There is an economy, $\mathcal{E}$, described by $n$ heterogeneous agents with von Neumann–Morgenstern expected utility, $U_i$, who are risk averse (Assumption 1) and have incomes at dates 1 and 0 which are subject to a common type of shock, with probability $p$ and of magnitude $L$ (Assumption 2). There are nontraded private insurance contracts, $\mathcal{I}$, which can be bought only at date 0. There are also financial assets, $\mathcal{D}$: a bond and shares of $J$ insurance companies, which can be traded at any date, $t \in [0, 1]$. Prices of insurance contracts, $S_i^I$, are determined by competition, while share prices, $S_j(t)$, are determined by no-arbitrage. Share prices change over time as new information on the final value of insurance company shares enters the stock market.

For this economy, agent's consumption, insurance, and investment decisions together with asset and insurance prices form a competitive insurance market equilibrium if agent's consumption allocations, $x_i$, resulting from their investment and insurance decisions are optimal, given their budget constraints ($B_i(\mathcal{D}, S_i^I)$):

[9] A formal and detailed description of $B_i(\mathcal{D}, S_i^I)$ is included in the Appendix.

**Definition 2:** *A triple* $((x_i)_{i=1}^n, (S_i^I)_{i=1}^n, \mathcal{D})$ *is a competitive insurance market equilibrium if:*

*(i)* *there exists* $(Q, r)$ *such that every* $S_j \in \mathcal{D}$ *satisfies no-arbitrage (Relation (1)),*

*(ii)* *insurance prices are competitive, and*

*(iii)* *for all* $i = 1, \ldots, n, x_i \in B_i(\mathcal{D}, S_i^I)$ *and for all* $x' \in B_i(\mathcal{D}, S_i^I), U_i(x_i) \geq U_i(x')$.

## EFFICIENCY AND EQUILIBRIUM WITH AN INSURANCE MARKET

This section studies the properties of a competitive insurance market equilibrium in our model. A key property of a financial market is whether it is complete or not. A financial market is *complete* if the set of available assets is sufficient to reproduce every possible state-contingent consumption allocation. If assets are sufficient to attain every Pareto efficient consumption allocation (but not every possible allocation) then the market is said to be *effectively complete*. In this section we establish the existence of efficient competitive insurance equilibria in which consumers attain the same allocations as they would if markets were complete.

The first notions of equilibria for financial markets are due to Arrow (1964) and Radner (1972). An important question to ask of such equilibria is: when are the allocations from a financial market equilibrium the same as those from a standard (Arrow-Debreu state-contingent commodity) equilibrium? As standard equilibria are Pareto efficient, a financial market equilibrium with the same allocations will also be efficient. The answer usually rests on determining whether financial markets are (effectively) complete.

When considering competitive insurance market equilibria, this article shows that the combination of private insurance with two dynamically traded assets generates effectively complete markets. In particular, the article shows that (1) for every state-contingent commodity equilibrium there is a corresponding competitive insurance market equilibrium that decentralizes it, and (2) in order to decentralize the state-contingent commodity equilibrium, there need only be two dynamically traded financial assets. These two dynamically traded assets are: a riskless bond and a fully diversified portfolio (described in the subsection. "Diversified Portfolios" on p. 321). The fully diversified portfolio can be as simple as an equally weighted portfolio of insurance company shares.

**Theorem 1:** *For every state-contingent commodity equilibrium of economy* $\mathcal{E}$ *with allocations* $((x_i^*)_{i=1}^n)$, *there exists a competitive insurance market equilibrium with the same allocations,* $((x_i^*)_{i=1}^n, (S_i^I))_{i=1}^n, \mathcal{D})$, *and where* $\mathcal{D}$ *contains only two assets: a zero coupon bond and a fully diversified portfolio.*

The proof of this result is based on the following chain of reasoning: every state-contingent commodity equilibrium is Pareto efficient; every Pareto efficient equilibrium of $\mathcal{E}$ implies optimal risk sharing; optimal risk sharing implies that consumption allocations have the mutuality property (described in the subsection "Diversified Portfolios" on p. 321); such consumption allocations can be constructed by purchasing full insurance, which eliminates risk from agents' endowments, and reallocating wealth net of insurance by dynamically trading the bond and the risky portfolio. To conclude, it is shown that the suggested combination of insurance and dynamic trading is feasible and optimal for every agent.

Theorem 1 implies:

**Remark 1:** Insurance markets can function efficiently when insurance firms are stock companies, although those stock companies have to be fully assessable (as Lloyd's syndicates used to be).

The assumption that stock companies are fully assessable is implicit in the definition of equilibrium as equilibrium imposes that the dividends promised by insurance companies are paid/collected in full. The flexibility in modeling the underlying risk in Assumption 3 allows one to have correlated date one risks and hence to apply this model to natural disasters.

Competitive insurance market equilibria that decentralize state-contingent commodity equilibria (as in Theorem 1) will be referred to as *efficient insurance market equilibria*. As this article makes repeated use of the combination of the bond and the portfolio of equally weighted insurance company shares, we will denote their prices by $\mathcal{D}^*$.

Given that a state-contingent commodity equilibrium for $\mathcal{E}$ always exists, then it follows that:

**Corollary 1:** *There exists* $(Q, r)$ *and* $(x_i)_{i=1}^n$ *such that* $((x_i)_{i=1}^n, (S_i^I)_{i=1}^n, \mathcal{D}^*)$ *is an efficient insurance market equilibrium of* $\mathcal{E}$.

Note that financial markets trade the shares of the $J$ insurance companies. Therefore, the actual securities traded, $\mathcal{D}$, is different from $\mathcal{D}^*$. But, even though the equally weighted portfolio is not traded, it may be synthesized using existing assets.[10] In such cases we say that $\mathcal{D}$ contains $\mathcal{D}^*$. Then, in the economy where only the bond and individual insurance company shares are traded, Theorem 1 says that an optimal investment strategy for any agent is to construct equally weighted portfolios.

## DEMAND FOR FULL INSURANCE

In the previous section, it is established that an economy in which an equally weighted portfolio of insurance company shares and a riskless bond are traded dynamically has an efficient insurance market equilibrium. This section presents the central result of the article: insurance agents will buy full coverage in this equilibrium. It is also shown that in equilibrium prices of insurance will be unfair, i.e., above their actuarial value. This seemingly contradictory result will be analyzed and explained in detail in the following section ("Explaining Full Insurance With Unfair Prices").

### Full Insurance in Equilibrium

In order to determine how much insurance agents are buying in equilibrium, one needs to make sure that the question can be properly framed. The model in its full

[10] Normally, a derivative asset (like a put or a call) is synthesized by dynamically trading two (or more assets). To synthesize a European call on an asset such as IBM stock one could build a portfolio with the asset (IBM stock) and a riskless bond (or money market account). Then, as the price of the asset changed over time one would need to rebalance the portfolio by buying and selling the asset. Synthesizing the equally weighted portfolio, on the other hand, is much easier, as once the portfolio is constructed the first time one does not need to rebalance it over time.

generality includes economies for which the demand for insurance is not uniquely defined by agents' purchases of private insurance contracts. For example, there could be only two agents and two insurance companies in the economy. If each insurance company insures a different agent it would be impossible to separate insurance coverage decisions from investment decisions as the dividends from the insurance company are perfectly (negatively) correlated with the indemnity payments received by one of the agents. An additional unnatural characteristic of such an economy is that agent's private insurance contracts are explicitly tradable (although labeled as "shares of insurance company X"). To avoid these extreme cases, assume that insurance contracts are *strictly nontradable*: insurance contracts in economy $\mathcal{E}$ are said to be strictly nontradable if there is no way of constructing a portfolio of existing traded assets that acts as a private insurance contract for any agent in the economy.

**Definition 3:** *Insurance contracts in economy* $\mathcal{E}(\mathcal{I}, \mathcal{D})$ *are strictly nontradable if* $\forall i$, $\nexists (\theta_j)_{j=0}^J$ *such that* $\sum_{j=0}^J \theta_j d_j(1) = N_i(1)$.

Recall that, even though insurance firms are not liable to free entry, our assumptions on insurance pricing (the subsection "Insurance Pricing" on p. 321) require enough competition both between insureds and insurers, that is, a sufficiently large $J$ and $n$. In addition, a sufficient condition to ensure that insurance contracts are strictly nontradable is to assume that each insurance company is the exclusive insurer for at least two agents, so that $J \leq n/2$. These assumptions seem quite reasonable for modern economies with sophisticated insurance and stock markets, which are the ones this article addresses.

With the notion of nontradable insurance contracts we can establish the primary contribution of this article: the optimal insurance coverage decision can be uniquely determined and for every efficient insurance market equilibrium every agent will optimally choose full insurance coverage.

**Theorem 2:** *In every efficient insurance market equilibrium such that* $\mathcal{D}$ *contains at least a zero coupon bond and a fully diversified portfolio, and where insurance contracts in economy* $\mathcal{E}(\mathcal{I}, \mathcal{D})$ *are strictly nontradable, then full insurance coverage will be the unique optimal insurance coverage.*

### The Loading on Insurance

Economists, specially those of us interested in the economics of insurance, are familiar with Mossin's (1968) result that "if the [insurance] premium is actuarially unfavorable, then it will never be optimal to take full coverage." Thus, a reaction would be to think that insurance prices in an efficient equilibrium must be fair. On the other hand, it is well known that variations in aggregate wealth will lead to unfair prices in general equilibrium, and this is what we find here:

**Theorem 3:** *In every efficient insurance equilibrium, for every agent in the economy the price per unit of coverage is the same,* $\forall i \in IS_i^I = S$, *and this price has a strictly positive loading,* $\exists \gamma > 0$ *such that:*

$$S = p(1 + \gamma)e^{-\mathbf{r}}. \tag{3}$$

The positive loading comes from the presence of aggregate risk in the economy. Insurance companies need to convince investors to buy risky shares and the only way to do so is to promise them an expected return that is higher than the riskless rate. This extra return is paid for with the loading.

## EXPLAINING FULL INSURANCE WITH UNFAIR PRICES

First of all let us be clear: Mossin did not make a mistake. The main difference between the results in this article and Mossin's is the context in which the coverage decision is made. Mossin makes a statement about how agents behave when faced with an isolated insurance decision. In this article,on the other hand, agents make insurance decisions as well as other investment decisions. The question now is what makes the current setup special that leads to such different results.

The answer is that there are three conditions that are satisfied in equilibrium, and which together imply that agents' optimal demand for insurance will include full coverage. The conditions are that:

(i) insurance prices are competitive,
(ii) only aggregate risk matters, and
(iii) financial markets are effectively complete.

If these three conditions are satisfied, then the purchase of full insurance is the only way to equalize the marginal utility of consumption across the loss and no loss states, *conditional* on the aggregate state of the world. The first condition has been formally introduced earlier in the description of the model ("Insurance and the Stock Market"). The other two need more explanation.

### Equilibrium Prices

Condition 2, that only aggregate risk matters, addresses the properties of financial prices. It says that financial prices do not put a premium on idiosyncratic risks that should, in principle, be diversified away. Given that insurance prices are competitive and hence derived from financial markets (as described in Theorem 3), this condition says that the loading on insurance arises only from the undiversifiable component of the private insurance contract. Condition (ii) is stated formally as:

**Definition 4:** *In economy $\mathcal{E}$, a measure $Q$ is said to price only aggregate risk if there exists a real-valued function $g$ such that the Radon-Nikodim derivative $\xi(1) = g(e(1))$.*

**Proposition 1:** *For every state-contingent commodity equilibrium of $\mathcal{E}$ with state-contingent prices $\tilde{\pi}$, $(x_i^*, \tilde{\pi})$, there exists $(Q, r)$ derived from $\tilde{\pi}$ such that $Q$ prices only aggregate risk and for any asset with dividend $d_j$, there exists a price process $S_j(t)$ satisfying Relation (1), i.e.,*

$$\forall t \in [0,1], \quad S_j(t) = E_{Q_t}\left[d_j e^{-\int_t^1 r(s)\,ds}\right] \quad P\text{-a.s.}$$

Proposition 1 follows naturally from the representative agent representation of prices of every state-contingent equilibrium of $\mathcal{E}$.[11] Suppose that one is presented with a particular state-contingent equilibrium of $\mathcal{E}$, where the representative agent's preferences are characterized by the utility function $V$:

$$V(x) = v_0(x(0)) + \beta_0 E_P[v_1(x(1))].$$

Then,

$$\xi(1) = \frac{v_1'(e(1))}{E_P[v_1'(e(1))]}. \tag{4}$$

### Effective Completeness and Optimal Investment

Condition (iii) addresses the individual's ability to diversify. In the economy $\mathcal{E}$ all agents are risk averse and hence would like to diversify. The third condition ensures that they have access to financial assets that will allow them to replicate any fully diversified portfolio. The formal statement of Condition (iii) is:

**Definition 5:** *In economy $\mathcal{E}$, financial markets are effectively complete if for every random dividend $d$ that satisfies the mutuality property, there exists $\theta \in \Theta$ such that*

$$d = \sum_{j=0}^{J} \theta_j(0) S_j(0) + \int_0^1 \sum_{j=0}^{J} \theta_j(t)\, \mathrm{d}S_j(t).$$

As Pareto efficient consumption allocations satisfy the mutuality property, a key step in the proof of Theorems 1 and 2 is to show that a bond and a fully diversified portfolio of insurance company shares (such as the equally weighted portfolio) make financial markets effectively complete. Rather than appeal to a generic result on effective completeness, we proceed by characterizing, using *deterministic* equations, an (stochastic) investment strategy that will replicate any fully diversified portfolio.

**Theorem 4:** *Suppose that in economy $\mathcal{E}$, the measure $Q$ prices only aggregate risk and there is a fully diversified portfolio with price process, $S_M(t)$, and dividend $d_M = g(e(1))$. For any date 1 dividend, $d$, that satisfies the mutuality property, i.e., $d = f(e(1))$, there exists deterministic functions $F_M(t,m), F_0(t,m), G_M^*(t,m), G^*(t,m), m = 0,1,\ldots,n, t \in [0,1]$, such that $d$ can be attained with an initial amount of money, $D^*(0)$, and by dynamically trading the bond and the fully diversified portfolio following the trading strategy:*

1. *For any $t \in [0,1]$, if $N(t-) = m < n$, invest $\theta_0 = F_0(t,\ m)$ is the riskless bond and $\theta_M = F_M(t,\ m)$ in the risky asset, where*

$$F_M(t,m) = \frac{G^*(t,m+1) - G^*(t,m)}{G_M^*(t,m+1) - G_M^*(t,m)}$$

$$F_0(t,m) = \frac{G^*(t,m) - F_M(t,m)G_M^*(t,m)}{e^{-\mathbf{r}}}.$$

[11] See, for example, Constantinides (1982) or the proof of Proposition 1 in the Appendix.

2. *If* $N(t-) = n$*, invest* $\theta_0 = F(t, n) = G^*(t, n)e^{\mathbf{r}}$ *in the riskless bond.*

We describe the basic steps of the proof here and refer the reader to the Appendix for details. As $d$ satisfies the mutuality property, then $d = f(e(1))$, as noted. To attain $d$ one needs to start with a certain amount of money, use it to construct an investment portfolio, and follow a trading strategy that will ensure that the value of the portfolio will, in the end, be equal to $d$.

First, define two processes, $D^*(t)$ and $S^*_M(t)$, where $D^*(t)$ is the date 0 discounted value of $d$ given information known at date $t$, and $S^*_M(t)$ the date 0 discounted price of the fully diversified portfolio, whose dividends are also a function of the aggregate endowment, $d_M = g(e(1))$. Then, $D^*(0)$ will be a constant equal to the expected discounted value of $d$ at date 0 and also the initial amount of money needed to construct the portfolio. $D^*(t)$ gives us the value of the portfolio that the trading strategy has to track. The problem of tracking $D^*(t)$ can be split into two parts: one is maintaining the value of the portfolio and the other is making sure that any news that change $D^*(t)$ (and security prices) will be matched by changes in the value of the portfolio so that it continues to track $D^*(t)$. The bond helps us with the first part, and the portfolio for the second.[12] Recall that $e(t)$ is a linear function of $N(t)$. Then, as $N(t)$ jumps randomly (whenever an agent has an accident $N(t)$ jumps up by one unit), $D^*(t)$ and $S^*_M(t)$ will also jump (though not necessarily by one unit). These processes, $D^*(t)$ and $S^*_M(t)$, can be represented in terms of the deterministic functions, $G^*(t, m)$ and $G^*_M(t, m)$ mentioned in Theorem 4 and described in detail in the Appendix ($D^*(t) = G^*(t, N(t))$ and $S^*_M(t) = G^*_M(t, N(t))$).

We now determine the units of the diversified portfolio needed to track $D^*(t)$. Using the fact that $D^*(t)$ and $S^*_M(t)$ can be described using the deterministic functions $G$ and $G^*_M$ we define the function $F_M(t, m)$. This function, $F_M$, describes the units of the diversified portfolio needed as a function of the date $t$ and the number of accidents, $m$, up to (but not including) date $t$. The amount invested in the diversified portfolio is chosen so as to match changes in the value of $D^*(t)$ if there is a new accident:

$$F_M(t, m) = \frac{G^*(t, m+1) - G^*(t, m)}{G^*_M(t, m+1) - G^*_M(t, m)}.$$

Then, the number of units invested in the riskless asset, $F_0(t, m)$, are chosen so as to make sure the value of the portfolio continues to track $D^*(t)$ if there is no accident:

$$F_0(t, m) = \frac{G^*(t, m) - F_M(t, m)G^*_M(t, m)}{e^{-\mathbf{r}}}.$$

Note that the functions $F_M$ and $F_0$ are deterministic (depend only on two numbers $t$ and $m$). On the other hand, the trading strategy $(\theta_0, \theta_M)$, combines the functions $F_M$ and $F_0$ with the stochastic process $N(t)$, so that the strategy is stochastic. This resulting strategy will generate a tracking portfolio whose value will always be equal to $D^*(t)$ and hence will be equal to $d$ at date 1.

[12] That one portfolio of shares is enough to track the changes can be proven using a martingale representation theorem—the article provides an alternative, constructive proof.

Combining Theorem 4 with Proposition 1 one obtains that Condition (iii), that financial markets are effectively complete, is satisfied in equilibrium:

**Remark 2:** For every efficient insurance market equilibrium, $((x_i)_{i=1}^n, (S_i^I))_{i=1}^n, \mathcal{D})$, the combination of a riskless bond and fully diversified portfolio of insurance company shares make financial markets effectively complete.

### Optimality of Full Insurance

With these results all that is needed to complete the proof of Theorem 2 is to show that the above strategy is optimal and ensuring that no partial coverage decision will lead to an optimal consumption allocation. The next result takes that step:

**Theorem 5:** *In economy $\mathcal{E}$, with prices $(\mathcal{D}, S_i^I)$, agent i's optimal insurance demand includes buying full insurance if:*

- *(i) there exists $(Q, r)$ such that $\mathcal{D}$ satisfy Relation (1) and Q prices only aggregate risk;*
- *(ii) insurance prices, $(S_i^I)_{i=1}^n$, are competitive under $(Q, r)$; and,*
- *(iii) the set of assets $\mathcal{D}$ makes financial markets effectively complete.*

*If agent i's insurance contract is strictly nontradable then agent i's unique optimal insurance demand is to purchase full coverage.*

Theorem 5 establishes why in an efficient insurance market equilibrium the optimal insurance strategy is to purchase full insurance: Conditions (i) and (ii) make the price of insurance "economically fair"; i.e., it exactly compensates for the actuarial and the economic risk (the probability and magnitude of indemnity payments plus the market price for undiversifiable risk, minus the time value of premium payments), and Condition (iii) says that having taken care of their idiosyncratic risk by buying full insurance, agents can construct their preferred fully diversified consumption allocation by trading in financial markets. Optimality of insurance and the trading strategy described in Theorem 4 then follows from the fact that the budget constraint in the efficient insurance equilibrium is contained in the agent's budget constraint in the state-contingent commodity equilibrium.

Note that a trivial corollary of Theorem 2 is that every agent's optimal investment decision in an efficient insurance market equilibrium is independent of his demand for insurance—independence meant in the classical statistical sense: knowing the demand for insurance in any efficient equilibrium (full coverage) tells us nothing about the individual's optimal investment decision.

## DISCUSSION AND EXTENSIONS

### Insurance Equilibrium and Effective Completeness

In the section "Efficiency and Equilibrium With an Insurance Market," it was shown that efficient competitive insurance markets equilibria exist with only two dynamically traded assets plus private insurance. A useful illustration of the methodological contribution in this article is to use the notion of insurance market equilibrium to express existing results on optimal risk sharing and effective completeness with frictionless financial markets. This also helps illustrate the value of the contribution made by Theorem 1.

*Arrow Securities.* Arrow's pioneering work (Arrow, 1964) on the decentralization of state-contingent equilibria using financial assets demonstrates that in general complete markets requires at least one asset for each state of the world. In $\mathcal{E}$ (ignoring the uncertainty on the exact timing of accident arrivals) there are $2^n$ states of the world, one for each possible realization of $\mathbf{N}(1)$. As Arrow did not consider the possibility of dynamically traded assets, his financial equilibrium can be considered as a competitive insurance equilibrium where everyone has access to the same securities, $d_j^A$, and can freely contract on them, $\theta_j \in \mathbf{R}$, but (like insurance contracts) they are not traded dynamically: $\mathcal{I} = \mathcal{I}_i = \{(\theta_j d_j)_{\theta_j \in \mathbf{R}}\}_{j=1}^{2^n}$, and $\mathcal{D} = \emptyset$.

*Pure and Mutual Insurance.* Malinvaud (1973) studied Arrow's economy using insurance contracts rather than Arrow securities. He considers an economy where $\mathcal{I}$ is the set of private insurance contracts. He shows that even if all agents are the same (in terms of preferences, risks, and endowments) there is no efficient insurance market for $\mathcal{E}(\mathcal{I}, \emptyset)$ because individual risks generate aggregate risk and $n$ insurance contracts are insufficient to deal both with individual risk and aggregate risk. On the positive side, he shows that as $n$ goes to infinity, aggregate risk (or rather the amount of aggregate risk apportioned to each agent) diminishes down to zero (by the Law of Large Numbers) and an approximate equilibrium can be constructed using only the $n$ insurance contracts.

Cass et al. (1996) expand $\mathcal{I}$ to include not just private insurance but also a class of contracts they call "mutual insurance" (contracts that depend on $N_i(1)$ and $N(1)$). They study $\mathcal{E}(\mathcal{I}, \emptyset)$ and conclude that the number of assets needed to effectively complete the market can be reduced (down from Arrow's $2^n$). The number depends on the number of types of agents, i.e., on the amount of symmetry across agents in terms of risks AND endowments and preferences. More precisely, if $H$ is the number of types of agents in the economy $\mathcal{E}$, the number of different types of assets needed is $(n+1)H$ mutual insurance contracts plus $n+1$ Arrow-type securities (securities that pay only if a particular aggregate state is reached).[13]

*Dynamically Traded Assets.* Duffie and Huang (1985) take Arrow's model and allow dynamic trading of all assets ($\mathcal{I} = \emptyset, \mathcal{D} \neq \emptyset$). They demonstrate that if agents can change their asset positions over time, then the number of assets needed to complete the market can be much smaller than $2^n$. They determine that there exists a number $K$ such that agents need at most $K+1$ assets (which have to satisfy certain abstract properties). In the Appendix it is demonstrated that for economy $\mathcal{E}$, $K = n$, so that $\mathcal{E}(\emptyset, \mathcal{D})$ has complete financial markets if $\mathcal{D}$ contains $n$ appropriate financial assets.

[13] Note also, that if two agents $i$ and $j$ are of the same type of agent, the mutual contract that pays depending on the state of agent $i$ and the aggregate state $m$ is "the same" contract as that depending on the state of agent $j$ and aggregate state $m$; i.e., these two contracts are counted as one type of contract.

*Insurance and Dynamically Traded Assets.* Theorem 1 shows that state contingent equilibria can be decentralized with very few assets: $n$ insurance contracts and two financial assets are sufficient. Furthermore, each agent uses only three contracts: his private insurance and the two financial assets.

**Remark 3:** It is possible to decentralize a complete market equilibrium allocation of $\mathcal{E}$ with only **two** dynamically traded financial assets in the economy, plus private insurance for each agent.

## HARA Preferences and Linear Risk-Sharing Rules

An interesting special case of the model (mainly due to its extensive use in the literature) is when agents have HARA preferences. Then, agents' optimal investment strategy is to buy assets at date 0 and hold on to them (i.e., not trade them at all):

**Proposition 2:** *If agents' preferences in $\mathcal{E}$ are of the form*

$$U_i(x) = v_i(x(0)) + \beta_i E(u_i(x(1))),$$

*where $-u_i'(x)/u_i''(x) = a_i + bx$, and if agents have access to a bond, full insurance and an equally weighted portfolio of insurance company shares, then agents' optimal investment strategies are to buy and hold the bond and the equally weighted portfolio and purchase full coverage.*

This result takes advantage of the linearity of the optimal sharing rule and the fact that the dividends of the equally weighted portfolio are a linear function of the aggregate endowment. The result is proven by constructing the strategy explicitly (uniqueness of the optimal strategy is not guaranteed but the arguments used in previous sections apply here as well).

The proof combines several results. The outline of the proof is: (1) The aggregate endowment is a linear function of total loss: there exists $w$, such that $e(1) = w - N(1)\,L$. (2) Take a Pareto optimal allocation $(x_i)_{i=1}^n$. By Pareto optimality and HARA preferences, each consumer's optimal consumption allocation is linear in the aggregate endowment; i.e., for all $i = 1, \ldots, n$, there exists $\alpha_i$ and $\beta_i$ (which depend on the parameters of the model) such that $x_i(1) = \alpha_i + \beta_i e(1) = (\alpha_i + \beta_i w) - \beta_i N(1)L$. (3) In order to attain this optimal allocation, everyone buys full insurance, and hence total liabilities are also a function of total loss: if $d_M$ is the date 1 dividend of the equally weighted portfolio, there exists $k_0$, $k_1$ such that $d_M = k_0 - k_1 N(1)$. (4) The agent's optimal strategy is to construct a portfolio with a final dividend, $d = x_i(1) - w_{i,1}$ (see proof of Theorem 1 in the Appendix). Insurance prices, $S_i^I$, are obtained using the prices from the state-contingent commodity market equilibrium. Finally, the buy-and-hold investment strategy $(\theta_0, \theta_M)$ is the solution to the following simple linear system of equations:

$$\theta_0 + \theta_M (k_0 - k_1 N(1)) = (\alpha_i + \beta_i w) - \beta_i N(1)L - w_{i,1}$$

$$\theta_0 S_0(0) + \theta_M S_M(0) = w_{i,0} - x_i(0) - S_i^I L.$$

Because agents do not trade dynamically, agents can be exposed to multiple risks and multiple occurrences of the same accident and Proposition 2 would continue to hold verbatim.

### Future Research

A question that remains open is the possibility of the existence of *inefficient* competitive insurance market equilibria. This problem is not unique to the current model but is shared by any model which has effectively complete (but not complete) markets. The sufficient conditions established in the section "Explaining Full Insurance Unfair Prices" imply that for an inefficient equilibrium to exist when the two basic financial assets are present (bonds and a diversified portfolio of insurance company shares) financial prices have to make a distinction across individual risks, but the author is not aware of any results that would generally rule out such prices using only equilibrium conditions.

A final interesting question is what would happen if there were additional costs of capital, such as taxes or costs arising from allowing insurance companies that are not fully assessible. In our competitive setting, insureds pay all the costs (and only the costs) of the insurance activity. Additional capital costs, such as taxes (as in Harrington and Niehaus, 2003) or default risk[14] would trickle down to insurance premia and it would be interesting to determine what effect these will have on agents' insurance decisions.

## APPENDIX

The order of the results in the Appendix is slightly different from the text. This order makes the logic of the proofs more transparent. The first section ("Formal Definitions") contains basic mathematical definitions that were unsuitable for the presentation but are used in the proofs. The second section ("Martigale Dimension") proves the remark made in the text that the number of financial assets needed to decentralize a state-contingent commodity equilibrium, $K$, is equal to the number of agents at risk, $n$. The third section ("Competitive Insurance Market Equilibria and Insurance Coverage") includes the proofs of Theorem 1, 2, 4 and 5 together with some auxiliary lemmas. Finally, the fourth section ("Proof of Theorem 3") contains the proof of Theorem 3.

### Formal Definitions

The Inada conditions on an increasing function $u : \mathbf{R}_+ \to \mathbf{R}$ are: $\inf_x u'(x) = 0$ and $\sup_s u'(x) = +\infty$. Note that this condition is sufficient though not necessary for the results in the article—they are used to guarantee existence and representative agent characterization of prices. Existence and representative agent characterization of prices can be extended to economies with preferences that do not satisfy the Inada conditions in the standard way.

*Stochastic Processes.* There is a canonical probability space on which the stochastic process, $\mathbf{N}$, is defined and is denoted $\Omega'$ (for more details on jump processes see

[14] The main existence results for general equilibrium with default (Dubey, Geanakoplos, and Zame, 2000; Araujo, Pascoa, and Torres-Martinez, 2002; Geanakoplos and Zame, 2002) prove existence, but their detailed implications for insurance have not been explored to the best of my knowledge.

Brémaud, 1981). The information generated by **N** is formally described by the filtration $(\mathcal{F}_t)_{t\in[0,1]}$ generated by **N**. Let $\sigma(x)$ denote the sigma-algebra generated by the random variable $x$, then for each $t \in [0,1]$, $\mathcal{F}_t = \cap_{s\leq t}\sigma(\mathbf{N}(s))$. A process $x(t)$ is said to be adapted to $(\mathcal{F}_t)_{t\in[0,1]}$ if for all $t$, $x(t)$ is $\mathcal{F}_t$-measurable. A process $x(t)$ is said to be $\mathcal{F}_t$-predictable if $x(t)$ is measurable with respect to $\mathcal{F}_{t-} \equiv \cap_{s<t}\sigma(\mathbf{N}(s))$. Let $x(t-)$ denote $\lim_{s\uparrow t} x(s)$. A process $x(t)$ is said to be $(P, (\mathcal{F}_t)_{t\in[0,1]})$-integrable if $x(t)$ is measurable with respect to $\mathcal{F}_{t-}$ and for all $t$ $\int |x(t)|P(\mathrm{d}\omega) < \infty$.

*Allowable Trading Strategies.* In an economy with $J$ assets, $\mathcal{D} = ((S_j(t))_{t\in[0,1]})_{j=0}^J$, let $\Theta$ denote the vector of allowable trading strategies given $\mathcal{D}$. Feasible trades satisfy the usual restrictions: an allowable trading strategy on asset $j$ is an $\mathcal{F}_t$-predictable and $(P, \mathcal{F}_t)$-integrable stochastic process $\theta_j^i(t)$, where $\theta_j^i(t)$ is the number of units of asset $j$ agent $i$ plans to hold going into date $t$. As real activity (endowments and consumption) takes place only at dates 0 and 1, feasible trades also require that allowable trading strategies satisfy the following self-financing condition: for all $\theta \in \Theta$:

$$\sum_{j=0}^{J}\theta_j(t)S_j(t) = \sum_{j=0}^{J}\theta_j(0)S_j(0) + \sum_{j=0}^{J}\int_0^t \theta_j(s)\,\mathrm{d}S_j(s), \quad \forall t \in [0,1].$$

*The Budget Constraint.* The budget constraint is the set of consumption allocations the agent can achieve given his endowment (and trading opportunities). In a competitive insurance market, agents can only alter their consumption by buying insurance and trading financial assets so that the set of attainable consumption allocations is determined by asset prices, $\mathcal{D} \equiv ((S_j(t))_{t\in[0,1]})_{j=0}^J$, allowable trading strategies, $\Theta$ (as defined above), and the availability of private insurance.

Keeping the model simple, the agent's initial asset holdings are assumed not to affect his total wealth (so that the distribution of wealth is uniquely determined by agent's risky income). Thus, if any agent has an initial endowment of insurance company shares, either the price of the shares is zero (which is the case if insurance companies are fully assessible and only issue equity) or any initial share holdings are compensated with a corresponding negative holding of the bond (i.e., initial share holdings are fully leveraged). Let $\theta_{j,0}^i$ denote the number of units of asset $j$ agent $i$ is endowed with at date 0. The complete description of agent $i$'s endowment is $e_i(0) = w_{i,0} + \sum_{j=0}^J \theta_{j,0}^i S_j(0), e_i(1) = w_{i,1} - N_i(1)L$.

Agent $i$'s budget constraint, $B_i(\mathcal{D}, S_i^I)$, is determined by asset prices, the price of his insurance coverage for him, $S_i^I$, and the set of allowable trading strategies $\Theta$:

$$B_i(\mathcal{D}, S_i^I) = \left\{ x = (x(0), x(1)) \;\middle|\; \begin{array}{l} \exists\theta^i \in \Theta,\ \alpha_i \in [0, L] \\ x(0) = e_i(0) - \alpha_i S_i^I - \sum_{j=0}^{J}\theta_j^i(0)S_j(0) \\ x(1) = e_i(1) + \alpha_i N_i(1) + \sum_{j=0}^{J}\theta_j^i(1)\,d_j \end{array} \right\}$$

### Martingale Dimension

Duffie and Huang (1985) show how to decentralize a state-contingent equilibrium as a Radner equilibrium if you have one riskless asset plus $K$ appropriate risky assets—we refer the reader to the original for full details, such as the exact definition of "appropriate." The number $K$ is equal to the dimension of the space of $(Q, (\mathcal{F}_t)_{t\in[0,1]})$-martingales, where the measure $Q$ is derived from the equilibrium price of the state-contingent commodity equilibrium via the Radon–Nikodym derivative, $\xi(1)$ (see the discussion after Relation (1) in the main text).

As $\xi(1) > 0$ $P$-a.s., then $Q$ is absolutely continuous relative to $P$. Fortunately, the martingale dimension of the space of martingales is invariant to an absolutely continuous change of measure so that to prove the claim in the text (that $K = n$) it suffices to show:

**Lemma 1:** *The space of martingales on* $(\Omega', \mathcal{F}_1, (\mathcal{F}_t)_t, P)$ *has martingale dimension of* $n$.

**Proof:** This follows from the properties of $M_i(t) \equiv N_i(t) - \int_0^t \lambda_i(s)\,ds$. Namely, the stochastic processes $M_1, \ldots, M_n$ are $(P, \mathcal{F}_t)$-martingales and pairwise orthogonal. Thus, applying the martingale representation theorem for marked point processes (see Last and Brandt, 1991, pp. 342–346), the martingales represent a *minimal* basis (if you eliminate any $M_i$, the set $\{M_1, \ldots, M_n\}\backslash\{M_i\}$ is not a basis) for the space of martingales so that the martingale dimension $K = n$. □

### Competitive Insurance Market Equilibria and Insurance Coverage: Proofs of Theorems 1, 2, 4, and 5 and Proposition 1

We proceed as follows:

1. Determine the properties of a state-contingent equilibrium;
2. Relate state-contingent equilibria with the Radon-Nikodym derivative and Condition (ii) (Proposition 1);
3. Determine effective completeness (Theorem 4);
4. Establish the insurance equilibrium (Theorem 1).
5. Establish the uniqueness of full insurance demand (Theorem 2);
6. Relate Condition (ii) and the mutuality property of optimal consumption;
7. Relate the optimality of full insurance to Conditions (i)-(iii) (Theorem 5);

§1. The main properties of state-contingent equilibria for this economy are summarized in the following result (see Constantinides, 1982):

**Proposition 3:** *A state-contingent equilibrium of* $\mathcal{E} = ((e_i, U_i)_{i=1}^n)$ *exists. Every state-contingent equilibrium of* $\mathcal{E}$, $((x_i^*)_{i=1}^n, \bar{\pi})$, *is Pareto efficient and there exists a representative agent representation of prices with strictly concave von Neumann–Morgenstern preferences*

$$v_0(x(0)) + \beta_0 E_P[v_1(x(1))]$$

*such that*

$$\tilde{\pi} = \frac{\beta_0 v_1'(e(1))}{v_0'(e(0))} \tag{A1}$$

*where $v_t'(x)$ is the first derivative of the representative agent's utility function at date $t = 0, 1$. For all $i$, there exists $f_i\colon \mathbf{R} \to \mathbf{R}$ such that $x_i^*(1) = f_i(e(1))$.*

§2. **Proof of Proposition 1 (in the "Equilibrium Prices" section):** Given a state-contingent equilibrium as described in Proposition 3 above, construct $Q$ using Equation (4). Then, let $e^{-\mathbf{r}} = E_P\,[v'_1(e(1))]$ and use it to define $r(t)$ using

$$S_0(t) = e^{-\mathbf{r}(1-t)} \equiv \exp\left(-\int_t^1 r(s)\,\mathrm{d}s\right).$$

Then, for any asset with dividends described by $d_j$ define $S_j(t) = E_{Q_t}[d_j S_0(t)]$. □

§3. **Proof of Theorem 4:** The proof of this result makes use of Duffie and Huang's (1985) idea of applying martingale representation results to construct trading strategies. The deterministic functions used to describe the strategy arise from the Poisson-like properties of the information represented by the aggregate accident process, $N(t)$.

To establish that a dividend with the mutuality property can be attained using the strategy specified in the theorem, one first has to construct the functions $G^*(t, m)$ and $G_M^*(t, m)$. The following lemma will be crucial:

**Lemma 2:** *Assumption 3 implies there exists a function $\mathcal{P} : [0, 1] \times \{0, 1, \ldots, n\}^2 \to [0, 1]$ such that*

$$P(\{N(1) = m \mid N(t) = n, t\}) = \mathcal{P}(t, n, m). \tag{A2}$$

**Proof:** This follows from the characterization of the distribution function of arrival times by their hazard rates. $\mathcal{P}$ is constructed recursively. Let $R_m(s, t)$ be the cumulative hazard rate from the hazard rate $(n - m)\eta(t, m)$, $R_m(s, t) = \int_s^t (n - m)\eta(r, m)\,\mathrm{d}r$. Then define the function $H : [0, 1]^2 \times \{0, 1, \ldots, n\}^2 \to [0, 1]$ as follows:

$$H(s, t, k, m) = , \quad \forall s \geq t$$

$$H(s, t, k, m) = 0, \quad \forall k > m$$

$$H(s, t, m, m) = \exp(-R_m(s, t))$$

$$H(s, t, k, m + 1) = n - m) \int_s^t H(s, r, k, m)\eta(r, m) \exp(-R_{m+1}(r, t))\,\mathrm{d}r,$$

$$m = k, \ldots, n - 1$$

Then, define $\mathcal{P}$ using $H\colon \mathcal{P}(t, k, m) = H(t, 1, k, m)$. □

[Theorem 4] As $d$ and $d_M$ have the mutuality property, there exists $f$ and $g$ such that $d_M = g(e(1))$ and $d = f(e(1))$. Abusing notation slightly, and using the fact that for

$m = 0, \ldots, n$, $e(1)$ and $\xi(1)$ are constant on $\{N(1,\omega) = m\}$, define $f(m) = f(e(1))$, $g(m) = g(e(1))$, and $\xi(1, m) = \xi(1)$ where $e(1)$ and $\xi(1)$ are evaluated on the set $\{N(1,\omega) = m\}$, $m = 0, \ldots, n$. Then, using the $\mathcal{P}(t, n, m)$ function constructed in the proof of Lemma 2 define the functions

$$\xi(t,k,r) = \begin{cases} \dfrac{\xi(1,r)}{\sum_{m=k}^{n} \mathcal{P}(t,k,m)\xi(1,m)}, & k = 0,\ldots,n;\ r = k,\ldots,n; \\ 1, & r < k \end{cases}$$

$$G^*(t,k) = \left(\sum_{m=k}^{n} \mathcal{P}(t,k,m)\xi(t,k,m) f(m)\right) e^{-\mathbf{r}}, \quad k = 0,\ldots,n;$$

$$G_M^*(t,k) = \sum_{m=k}^{n} \mathcal{P}(t,k,m)\xi(t,k,m) g(m) e^{-\mathbf{r}}, \quad k = 0,\ldots,n;$$

Note that with these functions one can construct the processes $G^*(t, N(t))$ and $G_M^*(t, N(t))$ where $G^*(t, N(t)) = E_{Q_t}[e^{-\mathbf{r}}d] = D^*(t)$ and $G_M^*(t, N(t)) = e^{-\mathbf{r}t} S_M(t) = S_M^*(t)$.[15]

The strategy to attain $d$ at date 1 (as in Duffie and Huang, 1985) is to solve the stochastic differential equation:

$$\mathrm{d}D^*(t) = \theta_M(t)\, \mathrm{d}S_M^*(t), \tag{A3}$$

Using the above equations, the solution to this stochastic differential equation can be described using a deterministic function, $F_M(t, m) : [0, 1] \times \{0, 1, \ldots, n-1\} \to \mathbf{R}$, so that $\theta_M(t) = F_M(t, N(t))$. The function $F_M(t, m)$ is constructed as follows: as Equation (A3) has to hold and $F_M(t, N(t))$ has to be predictable (investment strategies cannot anticipate surprise changes in stock prices), the value of $F_M(t, N(t))$ will be determined by what happens at accident times. This is because, $F_M(t, N(t))$ solves

$$G^*(t, N(t^-) + 1) - G^*(t, N(t^-)) = F_M(t, N(t))\big(G_M^*(t, N(t^-) + 1) - G_M^*(t, N(t^-))\big)$$

Define the function $F_M$:

$$F_M(t,m) = \frac{G^*(t,m+1) - G^*(t,m)}{G_M^*(t,m+1) - G_M^*(t,m)}, \quad m = 0,1,\ldots,n-1$$

Then, to reconstruct $x$, define

$$F_0(t,m) = \frac{G^*(t,m) - F_M(t,m)G_M^*(t,m)}{e^{-\mathbf{r}}}, \quad m = 0,1,\ldots,n-1$$

[15] Note, that both processes are $Q$-martingales and, as they are functions of $N(t)$, they can be shown to come from a one-dimensional space of martingales using the same arguments used in the previous section.

Following the strategy proposed in the theorem and that is defined by $(F_0, F_M)$ and an initial amount of money $D^*(0)$, the value of the portfolio $F_0(t, N(t))e^{-\mathbf{r}} + F_M(t, N(t))S^*_M(t)$ will be equal to $D^*(t)$ and hence equal to $d$ at date one. □

§4. **Proof of Theorem 1:** We need to construct a triple $((x_i)_{i=1}^n, (S_i^I))_{i=1}^n, \mathcal{D})$ that satisfies the conditions in Definition 2.

(i) By Proposition 3, every state-contingent commodity equilibrium is efficient as has a representative agent representation, so that for every $i$, $x_i^*$ has the mutualization property. By Proposition 1, construct $(Q, r)$ using $\tilde{\pi}$ so that $\mathcal{D}$ satisfy Relation (1) and $Q$ prices only aggregate risk.

(ii) Let $S_i^I = E_Q[N_i(1)e^{-\mathbf{r}}]$.

(iii) For each $i$, we first need to show $x_i^* \in B_i(\mathcal{D}, S_i^I)$. Assume the agent buys full insurance. This changes $i$'s allocation from the risky $(w_{i,0}, w_{i,1} - N_i(1)L)$ to the riskless $(w_{i,0} - S_i^I, w_{i,1})$. Then, apply Theorem 4 using $d = x_i^*(1) - w_{i,1}$ (recall from Proposition 3 that $x^*(1)$ has the mutuality property). This makes $i$'s allocation equal to $(w_{i,0} - S_i^I - D^*(0), w_{i,1} + d) = (w_{i,0} - S_i^I L - D^*(0), x_i^*(1))$.

Now use the $(Q, r)$ constructed above to show that the new allocation is equal to $(x_i^*(0), x_i^*(1))$: as $x_i^*$ is in the state-contingent budget constraint then, using the definition of $Q$ and $r: x_i^*(0) + E_Q[x_i^*(1)]e^{-\mathbf{r}} = w_{i,0} + E_Q[w_{i,1} - N_i(1)L]e^{-\mathbf{r}}$. From equilibrium pricing of insurance, $S_i^I = E_Q[N_i(1)]e^{-\mathbf{r}}$. By construction, $D^*(0) = E_Q[de^{-\mathbf{r}}] = E_Q[x_i^*(1) - w_{i,1}]e^{-\mathbf{r}}$. So that,

$$\begin{aligned} w_{i,0} - S_i^I L - D^*(0) &= w_{i,0} - E_Q[N_i(1)]e^{-\mathbf{r}}L - E_Q\big[x_i^*(1) - w_{i,1}\big]e^{-\mathbf{r}} \\ &= w_{i,0} + E_Q\big[w_{i,1} - N_i(1)L\big]e^{-\mathbf{r}} - E_Q\big[x_i^*(1)\big]e^{-\mathbf{r}} \\ &= x_i^*(0), \end{aligned}$$

which implies $x_i^* \in B_i(\mathcal{D}, S_i^I)$.

Finally, we need to show that $\forall x_i \in B_i(\mathcal{D}, S_i^I),\ U_i(x_i) \le U_i(x_i^*)$. As $(Q, r)$ are derived from state-contingent prices, $\forall x_i \in B_i(\mathcal{D}, S_i^I), x_i(0) + E_Q[x_i(1)e^{-\mathbf{r}}] \le w_{i,0} + E_Q[(w_{i,1} - LN_i(1))e^{-\mathbf{r}}]$ so that $B_i(\mathcal{D}, S_i^I)$ is contained in the budget constraint in the state-contingent equilibrium. As $x_i^*$ was optimal in the state-contingent equilibrium, it is also optimal restricted to the smaller budget constraint, $B_i(\mathcal{D}, S_i^I)$. □

§5. **Proof of Theorem 2:** Let $((x_i^*)_{i=1}^n, (S_i^I)_{i=1}^n, \mathcal{D})$ be an efficient insurance market equilibrium. Consider the strategy used in the proof of Theorem 1 to show $x_i^* \in B_i(\mathcal{D}, S_i^I)$. Suppose instead that agent $i$ optimally chooses partial insurance coverage, i.e., $\alpha_i \in [0, 1)$ and there is strategy $(\theta_j(t))_{t\in[0,1]}$ such that $x_i^* \in B_i(\mathcal{D}, S_i^I)$. Then, $(\theta_j(t))_{t\in[0,1]}$ achieves the date 1 net trade, denoted $X(1)$: $X(1) = x_i^*(1) - w_{i,1} + (1-\alpha_i)L\mathbf{1}_{N_i(1)=1}$, i.e., $\sum_{j=0}^J \theta_j(1)d_j(1) = X(1)$. However, note that (by Theorem 4) one can also construct a trading strategy $(\hat{\theta}_j(t))_{t\in[0,1]})$ such that $\sum_{j=0}^J \hat{\theta}_j(1)d_j(1) = x_i^*(1) - w_{i,1}$. Taking the difference between the two strategies $(\hat{\theta}_j - \theta_j)$ and dividing by $(1-\alpha)L$ one obtains a new strategy whose dividend is equal to $N_i(1)$. But, as insurance is strictly nontradable, such a trading strategy does not exist—a contradiction. □

§6. The next lemma shows that if Condition (ii) is satisfied, and markets are complete, optimal consumption satisfies the *mutuality property*:

**Lemma 3:** *For agents whose preferences satisfy Assumption 1, if there exists $(Q, r)$ such that $\mathcal{D}$ satisfy Relation (1) and the probability measure Q prices only aggregate risk then the optimal consumption in the following problem:*

$$\text{[Problem B]} \quad \max_x U_i(x) \text{ s.t. } \quad x(0) + E_Q[x(1)]e^{-\mathbf{r}} = e_i(0) + E_Q[e_i(1)]e^{-\mathbf{r}},$$

*has the mutuality property.*

**Proof:** Using $E_Q[z] = E_P[\xi(1)z]$, the Lagrangian for [Problem B] is

$$L = U_i(x) - \lambda\big(x(0) - e_i(0) + e^{-\mathbf{r}}E_P[\xi(1)(x(1) - e_i(1))]\big)$$

The necessary and sufficient first order conditions are:

$$v_i'(x(0)) = \lambda,$$

$$\forall\omega \in \Omega', \beta u_i'(x(1,\omega)) = \lambda\xi(e(1,\omega))e^{-\mathbf{r}},$$

where $\lambda$ is the Lagrange multiplier in the constrained maximization problem. The properties of $u_i$ ensure that $(u_i')^{-1}$ is a well-defined function so that the optimal consumption allocation, $x^*$, at date 1 is equal to $f_i(e(1))$ where:

$$f_i(e(1)) \equiv (u_i')^{-1}\left(\frac{v_i'(x^*(0))\xi(e(1))e^{-\mathbf{r}}}{\beta}\right)$$

and $x_i^*(0)$ is the constant that solves

$$x_i^*(0) + E_P[\xi(e(1))f_i(e(1))]e^{-\mathbf{r}} = e_i(0) + e^{-\mathbf{r}}E_P[\xi(1)e_i(1)].$$

The properties of the problem (preferences satisfying Assumption 1 plus the linearity of the constraint) imply that such a $x_i^*(0)$ exists. □

§7. **Proof of Theorem 5:** To prove this theorem one can follow a similar strategy as in the proof of Theorem 1: consider $(Q, r)$ given by Condition (i) and $S_i^I = E_Q[N_i e^{-\mathbf{r}}]$ from Condition (ii). Then, use Condition (iii) and Lemma 3 to determine that the the solution to [Problem B] is attainable using full insurance and asset trading. As the budget constraint in [Problem B] includes the budget constraint of an agent in economy $\mathcal{E}$ with prices $(\mathcal{D}, S_i^I)$, then full insurance could be agent $i$'s optimal insurance demand.

Uniqueness of full insurance coverage as the optimal insurance strategy follows by the same arguments used in the Proof of Theorem 2. □

## Proof of Theorem 3

The proof make use of the concept of exchangeability: for any arbitrary event $C$, let $\mathbf{1}_C$ denote the indicator function of $C$. The function $\iota : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$, is a permutation function on $\{1, \ldots, n\}$ if $\iota$ is bijective. The events $A_1, \ldots, A_n$ are exchangeable if for all permutations of the indexes, i.e., for all permutation functions on $\{1, \ldots, n\}$,

$$P(A_1, A_2, \ldots, A_n) = P\left(A_{\iota(1)}, A_{\iota(2)}, \ldots, A_{\iota(n)}\right).$$

Let $B_i$ represent an event of the kind $\{N_i(1) = 0\}$ (agent $i$ did not have an accident between dates 0 and 1) or $\{N_i(1) = 1\}$ (agent $i$ did have an accident between dates 0 and 1).

**Remark 4:** If $\lambda_i(t)$ satisfies Assumption 3 then any set of events $\{B_1, B_2, \ldots, B_n\}$ is a set of exchangeable events.

That this is true can be seen from the way the function $\mathcal{P}$ was constructed in the proof of Lemma 2.

**Proof of Theorem 3:** Let $A_i$ denote events of the kind $\{N_i(1) = 1\}$. Exchangeability implies that for $i, j, k \in \{1, \ldots, n\}$, $P(A_i) = P(A_k) = p$ and

$$\begin{aligned} P(N = j, N_i = 1) &= P(N = j, N_k = 1) \\ &= P(N = j) \binom{n-1}{j-1} \Big/ \binom{n}{j} = P(N = j)\frac{j}{n}. \end{aligned}$$

Let $q_i = E_Q[A_i]$, $\xi = \mathrm{d}Q/\mathrm{d}P$ and recall $n$ is finite, then

$$\begin{aligned} q_i &= \sum_{\omega \in A_i} P(\omega)\xi(\omega) \\ q_i &= \sum_{j=0}^{n} P(N = j \mid A_i)P(A_i)\xi(N = j) \\ q_i &= pE_P[\xi(N(1)) \mid A_i] \end{aligned}$$

and by exchangeability, $q_i = pE_P[\xi \mid A_k]$, $\forall\, k \in \{1, \ldots, n\}$. Thus, for all $i = 1, \ldots, n$, $q_i = q$. Let $p_j \equiv P(N = j)$, then

$$P(N = j \mid A_i) - P(N = j) = \frac{p_j}{p}\frac{j}{n} - p_j = \frac{p_j}{np}(j - np).$$

Also, $p = \sum_{k=0}^{n} P(N = k, A_i)$ and $P(N = k, A_i) = p_k k/n$, so that

$$P(N = j \mid A_i) - P(N = j) = \frac{p_j}{np}\left(j - n\sum_{k=0}^{n} p_k \frac{k}{n}\right)$$
$$= \frac{p_j}{np}(j - E[N]).$$

As $E[(j - E[N])] = 0$ and $N$ is increasing, then for all $k \leq n$

$$F_{N|A_i}(k) \equiv \sum_{j=0}^{k} P(N = j|A_i) \leq F_N(k) \equiv \sum_{j=0}^{k} P(N = j)$$

and the inequality is strict at least for $N = 0$. That is, $N(1) \mid A_i$ first-order stochastically dominates[16] $N(1)$. The economy has a representative agent representation with strictly increasing and concave utility (see Proposition 3) so that the equilibrium $\xi(N)$ will be strictly increasing. By definition $E[\xi] = 1$. Stochastic dominance of $F_{N|A_i}$, $\xi$ increasing, and $E_P[\xi] = 1$ imply $E_P[\xi \mid A_i] > 1$ and $q > p$. Putting everything together:

$$S_i^I = E_Q[N_i(1)]e^{-\mathbf{r}} = qe^{-\mathbf{r}} = p(1 + \gamma)e^{-\mathbf{r}}$$

with $\gamma > 0$. □

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[16] Given two real-valued random variables $X$ and $Y$ with cumulative distribution functions $F$ and $G$ respectively, $X$ dominates $Y$ in the first-order sense if $F(z) \leq G(z)$ for all $z \in \mathbf{R}$.

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