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SIGNAL ORDERINGS BASED ON DISPERSION AND THE SUPPLY OF PRIVATE INFORMATION IN AUCTIONS

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SIGNAL ORDERINGS BASED ON DISPERSION AND THE SUPPLY OF PRIVATE INFORMATION IN AUCTIONS

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This paper provides a novel approach to ordering signals based on the property that more informative signals lead to greater variability of conditional expectations. We define two nested information criteria (supermodular precision and integral precision) by combining this approach with two variability orders (dispersive and convex orders). We relate precision criteria with orderings based on the value of information to a decision maker. We then use precision to study the incentives of an auctioneer to supply private information. Using integral precision, we obtain two results: (i) a more precise signal yields a more efficient allocation; (ii) the auctioneer provides less than the efficient level of information. Supermodular precision allows us to extend the previous analysis to the case in which supplying information is costly and to obtain an additional finding; (iii) there is a complementarity between information and competition, so that both the socially efficient and the auctioneer's optimal choice of precision increase with the number of bidders.

KEYWORDS: Information, auctions, competition, variability orderings.

1. INTRODUCTION

THERE ARE NUMEROUS SITUATIONS in which a seller controls the information available to potential buyers: a government agency soliciting bids to execute a public project, a company wanting to sell a subsidiary (or go public), internet auctions, and so forth. Such situations raise important questions, such as should the seller make information available to buyers at all? How much information should he make available? Are his incentives to provide information aligned with social ones? How does his choice depend on the number of potential buyers in the market?

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We address these questions in an auction setting. To analyze the auctioneer's information supply problem generally, without recourse to specific families of signals, we need a criterion of informativeness. Orderings based on the value of information for a decision maker, such as Blackwell (1951) and Lehmann (1988), are well suited for analyzing the incentives of bidders to acquire information before making their bids. However, the seller faces a different kind of problem when deciding on the information available to buyers. A seller is not primarily interested in improving buyers' decisions, but in how the information provided affects the distribution of buyers' expected valuations and his profits.

The first contribution of this paper is to propose a new family of precision criteria which are especially tailored to such situations where one wishes to measure information by its impact on the distribution of conditional expectations. *Precision criteria* are defined based on the property that more informative signals lead to a more disperse distribution of conditional expectations. This dispersion effect arises because the sensitivity of conditional expectations to the realized value of the signal depends on the informational content of the signal. If the informational content of the signal is low, conditional expectations are concentrated around the expected value of the prior. When the informational content is high, conditional expectations depend, to a large extent, on the realization of the signal, which increases their variability.

Formally, our approach is that an information structure, that is, the joint distribution of the state of the world and the signal, is more informative (more precise) than another if it generates a more disperse distribution of conditional expectations. Of the existing variability orders, we select two (the dispersive order and the convex order) to define two nested notions of precision (supermodular and integral precision). We relate our precision criteria to standard orderings based on the value of information. We show that these orderings only imply integral precision, the weaker of our two criteria. We also show that, in general, supermodular precision does not imply any of the standard information orderings. Hence, precision criteria are consistent with, but strictly different from, orderings based on the value of information.

Precision criteria allow us to obtain general results on the social incentives and the auctioneer's private incentives to supply information. We consider a setting where an auctioneer wishes to sell an object to n risk-neutral bidders who are initially uncertain about their valuations. Prior to the auction, each bidder receives a private signal which conveys information only about his private valuation. The auctioneer chooses these signals from a general class of signals ranked in terms of precision criteria. This choice can be interpreted either as the auctioneer producing information or as controlling access to existing information.

The first result is that greater precision raises the efficiency of the allocation. If bidders' signals are more precise, they generate a larger total surplus. Hence, if granting access to information is costless, then it is efficient to give full access to all available information.

The second result is that private and social incentives are not aligned. The auctioneer's expected revenue is total surplus minus informational rents of the winning bidder. Greater precision improves the efficiency of the allocation, which increases expected revenues while also increasing informational rents, which reduce them. Hence, informational rents increase the cost of providing information, and the auctioneer optimally chooses an inefficient level of precision.

With two bidders, the negative effect of information (increased informational rents) overwhelms the gains from a better allocation, and the value of providing information is negative. The auctioneer will not provide any information, even if to do so is costless. If the number of bidders is high enough, the efficiency effect dominates, and supplying information is valuable to the auctioneer. At the limit, when the number of bidders reaches infinity, informational rents disappear, and with them the difference between the optimal and the efficient levels of information disappears.

Therefore, we find that in contrast to the information unraveling results of Milgrom (1981) and Grossman (1981) and the linkage principle of Milgrom and Weber (1982), the seller cannot be relied upon to disclose all socially desirable information.[2] The reason is that private information increases differences between bidders, which generates informational rents and lowers the seller's incentives to provide information. This idea already appeared in Lewis and Sappington's (1994) pioneering study of information revelation by a monopolist. Lewis and Sappington (1994) used simple families of signals to show that the asymmetric reaction of buyers to private information may lead the monopolist to optimally withhold information. The current paper shows that this gap between private and social incentives to supply information, described in Lewis and Sappington (1994) (as well as in Che (1996), Ganuza (2004), Johnson and Myatt (2006), Bergemann and Pesendorfer (2007), and others), exists when signals are ranked in terms of integral precision, and hence for all standard notions of informativeness.

Supermodular precision allows us to extend the previous analysis to the case in which supplying information is costly. Furthermore, when signals are ordered in terms of supermodular precision, we show that information and competition are complements in the sense that total surplus and the auctioneer's expected revenue are supermodular in the number of bidders and the precision of the signal. This implies that the socially efficient and the auctioneer's optimal choice of precision are increasing in the number of bidders in the auction.

[2]Information unraveling typically refers to the situation where a seller is known to be informed about the quality of his product and can provide verifiable information at no cost. In this setting, given buyers' skeptical equilibrium beliefs, the seller can do no better than to disclose his information. In addition, the linkage principle applies to affiliated environments where, in contrast to our setting, buyers react symmetrically to the information provided by the seller.

2. RANKING SIGNALS ACCORDING TO THE VARIABILITY OF CONDITIONAL EXPECTATIONS

Let V be a random variable representing the unknown state of the world, and let X_k be a signal. The signal is defined by a family of distributions $\{F_k(x|v)\}_{v\in\mathbf{R}}$, where for each realization v of V, $F_k(x|v) = \Pr(X_k \leq x|V = v)$. Given a prior $H(v)$, the signal induces a joint distribution on (V, X_k), called an *information structure*.

Assume that V has a finite expectation μ and that $F_k(x|v)$ admits a density $f_k(x|v)$. The marginal distribution of X_k is denoted by $F_k(x)$ and satisfies $F_k(x) = \int^x \int_{\mathbf{R}} f_k(y|v)\, dH(v)\, dy$. Let $F_k(v|x)$ denote the posterior distribution of V conditional on $X_k = x$ and let $E_k[V|x]$ denote the conditional expectation of V given $X_k = x$. We assume that $E_k[V|x]$ is nondecreasing in x.[3] This expectation defines the random variable $E[V|X_k]$, with distribution $G_k(w) = \Pr\{x|E_k[V|x] \leq w\}$ and quantile function $G_k^{-1}(p) = \inf\{w|G_k(w) \geq p\}$.

2.1. *Precision Criteria*

For a given prior $H(v)$, we wish to compare a signal X_1 with another X_2 in terms of their informational content. We say X_1 is more precise than X_2 if $E[V|X_1]$ is more disperse than $E[V|X_2]$. We focus on two notions of dispersion.

DEFINITION 1—Univariate Variability Orders: Let Y and Z be two real-valued random variables with distributions F and G, respectively.

- *Dispersive Order*: Y is said to be greater than Z in the *dispersive order* ($Y \geq_{\text{disp}} Z$) if for all $q, p \in (0, 1)$, $q > p$,

$$F^{-1}(q) - F^{-1}(p) \geq G^{-1}(q) - G^{-1}(p).$$

- *Convex Order*: Y is greater than Z in the *convex order* ($Y \geq_{\text{cx}} Z$) if for all convex real-valued functions ϕ, $E[\phi(Y)] \geq E[\phi(Z)]$ provided the expectation exists.

If Y and Z have the same finite mean, then $Y \geq_{\text{cx}} Z$ if and only if Y is a mean-preserving increase in risk (MPIR)[4] of Z: Y is a MPIR of Z if $E[Z] = E[Y]$ is finite and for all $z \in \mathbf{R}$,

$$\int_{-\infty}^{z} F(x)\, dx \geq \int_{-\infty}^{z} G(x)\, dx.$$

[3] This assumption does not constrain the set of signals under consideration. For an arbitrary signal X' and prior $H(v)$, if $E[V|X' = x]$ is not monotone in x, a new, equivalent signal X can be defined by reordering the realizations of X' according to $E[V|x]$.

[4] The MPIR has been used extensively in modeling risk in economics as it characterizes second-order stochastic dominance for random variables with the same mean. Z second-order stochastically dominates Y if all risk-averse expected utility maximizers prefer Z to Y.

For variables with finite and equal means, these variability orders are nested: $Y \geq_{\text{disp}} Z \Rightarrow Y \geq_{\text{cx}} Z$ (Shaked and Shantikumar (2007, Theorem 3.B.16 and (3.A.32))).

Using Definition 1, we define two criteria to order signals in terms of their informativeness:

DEFINITION 2—Precision Criteria: Given a prior $H(v)$ and two signals X_1 and X_2, then:

(i) X_1 is more *supermodular precise* than X_2 if $E[V|X_1]$ is greater in the dispersive order than $E[V|X_2]$.

(ii) X_1 is more *integral precise* than X_2 if $E[V|X_1]$ is greater in the convex order than $E[V|X_2]$.

Notice that signals are ordered for a given prior. The prior plays a crucial role in the definition, as $E[V|X_k]$ is computed using both the prior and the signal. Thus, precision criteria are defined as orders over the information structures (V, X_k).

As $E[E[V|X_k]]=\mu$ for $k=1, 2$, precision criteria, like their dispersion counterparts, are nested.

PROPOSITION 1: *Given a prior $H(v)$ and two signals X_1 and X_2, if X_1 is more supermodular precise than X_2, then X_1 is more integral precise than X_2.*

2.2. *An Alternative Characterization of Precision*

We define a new signal by applying the *probability integral transformation* to the original signal, $\Pi_k = F_k(X_k)$. The transformed signal is uniformly distributed on $[0, 1]$.[5] As any two transformed signals, Π_1 and Π_2, have the same marginal distribution, their realizations are directly comparable, regardless of the distributions of the original signals X_1 and X_2. Furthermore, comparing $\Pi_1 = \pi$ with $\Pi_2 = \pi$ is equivalent to comparing the original signals using quantiles, that is, comparing the realization $X_1 = x$ with the realization of $X_2 = y$, where $F_1(x) = \pi = F_2(y)$.

Let $W_k(\pi) = E[V|\Pi_k = \pi] = E_k[V|F_k^{-1}(\pi)]$ be the normalized conditional expectation function. Using $W_k(\pi)$, we can provide an alternative characterization of precision.

LEMMA 1: *Given a prior $H(v)$, and two signals X_1 and X_2, then*:

[5] Π_k is uniform on $[0, 1]$ only if $F_k(x)$ is continuous and strictly increasing. This can be assumed without loss of generality. If F_k has a discontinuity at x, where $\Pr(X_k = x) = p$, X_k can be transformed into X^*, which has a continuous and strictly increasing distribution function using the following construction proposed in Lehmann (1988): $X^* = X_k$ for $X_k < x$, X^* is $X_k + pU$ if $X_k = x$, where U is uniform on $(0, 1)$, and $X^* = X_k + p$ for $X_k > x$.

(i) X_1 *is more supermodular precise than* X_2 *if and only if* $\forall\pi, \pi' \in (0, 1)$, $\pi > \pi'$, $W_1(\pi) - W_2(\pi) \geq W_1(\pi') - W_2(\pi')$.

(ii) X_1 *is more integral precise than* X_2 *if and only if* $\forall\pi \in (0, 1)$, $\int_0^\pi (W_1(p) - W_2(p))\,dp \leq 0$.

Lemma 1 follows from (a) the relationship between $W_k(\pi)$ and $G_k^{-1}(\pi)$, the quantile function of $E[V|X_k]$: $W_k(\pi) = G_k^{-1}(\pi)$, and (b) the definitions of dispersive order and MPIR (the latter characterized in terms of the inverse of the cumulative distribution; see Shaked and Shantikumar (2007, Theorem 3.A.5)).

Lemma 1 describes precision in terms of the sensitivity of conditional expectations to signal realizations, which is especially transparent in the case of supermodular precision. The more supermodular precise signal, X_1, has a conditional expectation function, W_1, that is steeper/more sensitive to changes in π than W_2 at every π.

2.3. *Comparing Precision With Other Information Orders*

Information orders based on the value of information are characterized by conditions that are necessary and sufficient for all decision makers with payoff functions in a particular class to prefer one signal over another: (i) Sufficiency (Blackwell (1951)) is the strongest condition since it applies to *all* decision makers; (ii) effectiveness (Lehmann (1988) and Persico (2000)) weakens Blackwell's criterion since it applies to decision makers with single-crossing preferences;[6,7] (iii) Monotone Information Order for Nondecreasing objective functions (MIO-ND, Athey and Levin (2001)) focuses on decision makers with supermodular preferences.[8] As single-crossing preferences include supermodular ones, MIO-ND is weaker than effectiveness.

MIO-ND and effectiveness are characterized on restricted domains. Effectiveness is characterized for signals that are monotone in the sense of Milgrom (1981), which is a more restrictive condition than that required to characterize MIO-ND. Hence, by restricting attention to monotone signals and the information structures they generate, we ensure that we are in a domain where all the above information orders are characterized.

[6]Effectiveness is characterized for monotone signals in the sense of Milgrom (1981), that is, those that satisfy the monotone likelihood ratio property (MLRP): X_k satisfies the MLRP if for all $x > x'$, $f_k(x|v)/f_k(x'|v)$ is nondecreasing in v. If X_1 and X_2 satisfy the MLRP, X_1 is more effective than X_2 if and only if for all x, $\gamma(v, x) = F_1^{-1}(F_2(x|v)|v)$ is nondecreasing in v.

[7]Persico (2000) proves that all decision makers with single-crossing preferences prefer one signal X_1 over another X_2 for all priors if and only if X_1 is more effective than X_2. Lehmann's original result is for decision makers with "KR-monotone" preferences. See Jewitt (2007) for a detailed discussion.

[8]The MIO-ND order is characterized for information structures (V, X_k) that satisfy $\forall v, F_k(v|x) \leq F_k(v|x')$. Given $H(v)$, if the previous condition is satisfied for $k = 1, 2$, X_1 is greater than X_2 in the MIO-ND order given $H(v)$ if for all v and $\pi \in (0, 1)$, $F_2(v|X \leq F_2^{-1}(\pi)) \leq F_1(v|X \leq F_1^{-1}(\pi))$.

Value-based information measures imply the following in terms of precision:

THEOREM 1: (i) *Given $H(v)$, if X_1 is greater than X_2 in the MIO-ND order, then X_1 is more integral precise than X_2.*

(ii) *There exists a prior $H(v)$, and two monotone signals X_1 and X_2 such that X_1 is sufficient for X_2, but X_1 and X_2 are not ordered in terms of supermodular precision.*

Theorem 1(i) implies that precision is consistent with standard notions of information. As MIO-ND is implied by the other value-based information orders, any two signals ordered in terms of their value will be equally ordered in terms of integral precision. This order can be lost, but not reversed, in terms of supermodular precision. Similarly, a signal that is more precise than another for a given prior may or may not be more valuable, but it will never be less valuable.

Theorem 1(ii) is illustrated with the following known models of information.

Partitions: Let $V \in H(v)$ with support on $[0, 1]$. Consider two signals generated by two partitions of $[0, 1]$, $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{B}$ is finer than $\mathcal{A}$. Using these partitions one can define signals X_1 and X_2 in the usual way: signal X_1 $[X_2]$ tells you which set in the partition $\mathcal{A}$ $[\mathcal{B}]$ contains v. X_2 is sufficient for and more integral precise than X_1, but X_1 and X_2 are not ordered in terms of supermodular precision. This is because, in general, $W_1(\pi)$ and $W_2(\pi)$ will cross more than once, and a necessary condition for supermodular precision is that they cross only once.

Uniform Experiments: Let $V \in H(v)$ with support equal to $[0, 1]$ and let $F_k(x|v)$ be uniform on $[v - 1/2k, v + 1/2k]$, where k is a given constant such that $k \geq 1$. Then X_k is sufficient for X_1 if $k \in \{2, 3, \dots\}$ (Lehmann (1988, Theorem 3.1)), while for any k, k', $k > k'$, X_k is more effective and more integral precise than $X_{k'}$. But X_k and $X_{k'}$ are not ordered in terms of supermodular precision. This is because extreme signals are equally informative: they reveal the underlying state of the world perfectly. This implies that $E[V|X_k]$ and $E[V|X_{k'}]$ have the same finite support, which precludes the dispersive order.

These examples suggest that supermodular precision may be stronger than standard informativeness criteria based on the value of information. Generally this is not the case, although below we consider one situation in which supermodular precision implies sufficiency.

PROPOSITION 2: *There exists a prior $H(v)$, and two monotone signals, X_1 and X_2, such that X_1 is more supermodular precise than X_2, but X_1 and X_2 are not MIO-ND ordered.*

2.4. *Precision in Applications*

2.4.1. *Precision and Dichotomies*

When the underlying state of the world is described by two values $V \in \{v_H, v_L\}$, $v_H > v_L$, with $\Pr(v_H) = q$, precision measures are easily characterized in terms of the properties of the transformed signal Π_k. For $j \in \{H, L\}$, let $\hat{F}_k(\pi|j) = \Pr(\Pi_k \leq \pi|V = v_j)$ with density $\hat{f}_k(\pi|j)$.

PROPOSITION 3: (i) *Given* $q \in (0, 1)$, X_1 *is more integral precise than* X_2 *if and only if* $\hat{F}_1(\pi|H) \leq \hat{F}_2(\pi|H)$ *for all* $\pi \in (0, 1)$.

(ii) *Given* $q \in (0, 1)$, X_1 *is more supermodular precise than* X_2 *if and only if* $\hat{f}_1(\pi|H) - \hat{f}_2(\pi|H)$ *is nondecreasing in* π.

Furthermore, integral precision is equivalent to Blackwell sufficiency (which is equivalent to effectiveness in the context of dichotomies; see Jewitt (2007)).

THEOREM 2: *Given* $q \in (0, 1)$, X_1 *is more integral precise than* X_2 *if and only if* X_1 *is sufficient for* X_2.

2.4.2. *Supermodular Precision With Linear Conditional Expectations*

The following proposition provides a sufficient condition for models with linear conditional expectations to be ordered in terms of supermodular precision.

PROPOSITION 4: *Given a prior* $H(v)$, *and two signals* X_1 *and* X_2, *if there exists a random variable* Z *such that for* $j = 1, 2$, $E[V|X_j] = \alpha(j) + \beta(j)Z$, *and if* $\beta(2) \geq \beta(1)$, *then* X_2 *is more supermodular precise than* X_1.

Normal Experiments: Let $V \sim \mathcal{N}(\mu, \sigma_v^2)$, for $V = v$, X_k is equal to $v + \epsilon_k$, where $\epsilon_k \sim \mathcal{N}(0, \sigma_k^2)$ and is independent of V, then $X_k \sim \mathcal{N}(\mu, \sigma_v^2 + \sigma_k^2)$. The variance of the noise term ϵ_k orders signals in the usual way: A signal with less noise (lower σ_k) is more informative in terms of supermodular precision. Let $Z \sim \mathcal{N}(\mu, 1)$ and rewrite $X_k = Z\sqrt{\sigma_v^2 + \sigma_k^2}$:

$$E[V|X_k] = \frac{\sigma_k^2}{\sigma_v^2 + \sigma_k^2}\mu + \left(\frac{\sigma_v^2}{\sigma_v^2 + \sigma_k^2}\right)X_k$$
$$= \frac{\sigma_k^2}{\sigma_v^2 + \sigma_k^2}\mu + \left(\frac{\sigma_v^2}{\sqrt{\sigma_v^2 + \sigma_k^2}}\right)Z.$$

A less noisy signal (lower σ_k) has a higher $\beta(k) = \sigma_v^2(\sigma_v^2 + \sigma_k^2)^{-1/2}$ and, by Proposition 4, is more supermodular precise.

The Linear Experiment: Let $V \sim H(v)$ with mean μ. With probability k, $X_k = V$, and with probability $1 - k$, $X_k = \epsilon$, where $\epsilon \sim H(v)$ and is independent of V. Let X_k and $X_{k'}$ be two such signals. Then $E_k[V|x] = (1-k)\mu + kx$. Let $Z \sim H(v)$ so that $E[V|X_k] = (1-k)\mu + kZ$. By Proposition 4, $k > k'$ implies X_k is more supermodular precise than $X_{k'}$.

Binary Experiment: Let $V = v_H$ with probability q and $= v_L$ with probability $1-q$. The signal X_k can take two values H or L. The probability $\Pr(X_k = H|v_H) = \frac{1}{2} + a\gamma_k$ and $\Pr(X_k = H|v_L) = \frac{1}{2} - b\gamma_k$, where $a, b, \gamma_k \geq 0$. A higher value of γ_k makes the signal more valuable as well as more integral precise. If $a = 1-q$ and $b = q$, then it is possible to apply Proposition 4 and show that γ_k will also order signals in terms of supermodular precision: With $a = 1-q$ and $b = q$, $\Pr(X_k = H) = 1/2$. Let $Z = 1$ with probability $1/2$ and $Z = -1$ with probability $1/2$. Then invoke Proposition 4 using

$$E[V|X_k] = \alpha + \beta\gamma_k Z, \quad \text{where}$$
$$\alpha = v_L + (v_H - v_L)b, \quad \beta = 2ab(v_H - v_L).$$

The binary experiment illustrates a difficulty encountered when applying the dispersive order (and hence, supermodular precision) to order discrete random variables. A necessary condition for a discrete random variable Y to be more disperse than another X is that the jumps in the quantile function of X coincide with jumps in the quantile function of Y. In the binary experiment this requirement translates to $\Pr(X_k = H) = \Pr(X_{k'} = H)$.

2.4.3. *Supermodular Precision With Copulas*

We provide a sufficient condition for supermodular precision that applies to information structures described using copulas. Given an information structure (V, X_k) with marginals $H(v)$ and $F_k(x)$ there exists a function, called the *copula*, $C_k(u, \pi)$, $C_k : [0,1] \times [0,1] \to [0,1]$, such that $\Pr(V \leq v, X_k \leq x) = C_k(H(v), F_k(x))$ (Sklar's theorem (Sklar (1959))). This copula is a joint distribution function with marginals that are uniform on $[0,1]$, and it captures the dependency between V and X_k independently of the marginals. For an up-to-date introduction to copulas, see Nelsen (2006).

PROPOSITION 5: *Given a prior $H(v)$, and two signals X_1 and X_2, such that for $k \in \{1,2\}$ and all v, x, $\Pr(V \leq v, X_k \leq x) = C_k(H(v), F_k(x))$. If for $u, \pi \in [0,1]$ the copula $C_k(u, \pi)$, $k \in \{1,2\}$, is differentiable in π and $-\partial C_k(u,\pi)/\partial\pi$ is supermodular in (k, π), then X_2 is more supermodular precise than X_1 given $H(v)$.*

An example of a family of copulas that satisfies this condition is the Farlie–Gumbel–Morgenstern family, defined for each $k \in [-1, 1]$ by

$$C(u, \pi; k) = u\pi + ku\pi(1-u)(1-\pi), \quad u, \pi \in [0,1].$$

2.4.4. *The Role of the Prior*

Precision criteria are defined for a given prior. Thus, it could occur that one statistical experiment generates greater precision than another with one prior, but that the ranking may not hold if another prior is used. Then, if there is uncertainty about the prior, to say that one signal is more precise than another, one would have to ensure that the precision ranking holds for all possible priors under consideration.

However, using Theorem 1(i) and the fact that greater effectiveness implies greater MIO-ND, we can obtain a sufficient condition for two signals X_1 and X_2 to be ordered in terms of integral precision for all priors.[9]

COROLLARY 1: *Given two monotone signals X_1 and X_2, if X_1 is more effective than X_2, then X_1 is more integral precise than X_2 for all priors.*

For example, consider the normal experiment described above with signals X_1 and X_2 with $\sigma_1 < \sigma_2$. If the prior is known to be normal, albeit with uncertainty over its mean and variance, X_1 is more supermodular precise than X_2. However, when the prior is concentrated on two values, $\{v_L, v_H\}$, there are parameter values such that X_1 and X_2 are not ordered in terms of supermodular precision.[10] Nevertheless, X_1 is more effective than X_2, so that by Corollary 1, X_1 is more integral precise than X_2 for all priors. Furthermore, X_1 and X_2 may or may not be ranked under the supermodular order, but the ranking will never be reversed, so there cannot exist a prior such that X_2 is more supermodular precise than X_1.

3. PRECISION AND THE SUPPLY OF PRIVATE INFORMATION IN AUCTIONS

3.1. *The Setup*

An auctioneer wishes to sell an object he values at zero, to one of $n \geq 2$ (ex ante) identical risk-neutral bidders (indexed by $i = 1, \dots, n$). Bidders' valuations of the object are private and uncertain. Bidder i's realized valuation after the auction is described by a random variable V^i. For all $i = 1, \dots, n$, V^i is independently distributed on $[0, 1]$ according to a common distribution $H(v) = \Pr(V^i \leq v)$ with mean μ. All bidders start with identical priors, described by $H(v)$ (this is common knowledge to all agents). Hence, their expected valuations of the object will be the same and equal to μ. The utility

[9]A direct proof of the connection between effectiveness and integral precision in the context of dependence orders is found in Mizuno (2006).

[10]Supermodular precision fails for reasons similar to those in the uniform experiment, where a constant finite support of $E[V|X_k]$ precludes the dispersion order: for extreme realizations of the signal, the difference between $W_1(\pi)$ and $W_2(\pi)$ is close to zero (as conditional expectations are close to v_H or v_L), while for intermediate values of π, the difference is greater and, hence, $W_2(\pi) - W_1(\pi)$ is not monotone.

obtained by bidder i from winning the auction if the realized valuation is v^i and he makes a monetary payment of t^i is

$$u^i(v^i, t^i) = v^i - t^i.$$

The auctioneer can supply information prior to the auction. The production of information is costly. By paying an amount $\delta \in [0, \infty)$, the auctioneer will generate information in the form of private signals $(\Pi_\delta^i)_{i=1}^n$.[11] The choice of δ is publicly observed by all bidders and determines the precision of the signals, which is the same for all bidders.[12] Signals are independent and identically distributed random variables. For each $i = 1, \dots, n$, Π_δ^i is informative only about bidder i's true and uncertain valuation v^i, and bidder i observes the private signal Π_δ^i and no other. We assume without loss of generality that the marginal distribution of Π_δ^i is uniform on [0, 1]. We omit sub- and superscripts whenever they are clear from the context.

After the auctioneer has released the information, the awarding process takes place. To participate in this process, each bidder combines his knowledge of δ and the realization of the private signal, π^i, and updates his expected valuation of the object to $W_\delta(\pi^i) = E[V^i | \Pi_\delta^i = \pi^i]$. Finally, the auctioneer sells the object using a second-price sealed-bid auction. We abstract from reserve prices and assume the object is always sold.[13]

We will assume that signals are ordered by δ in terms of supermodular precision. Nevertheless, whenever possible, results are stated for integral precision. In Section 3.5, we review the results with integral precision and their implications for the case with costless provision of information.

3.2. *The Efficient Release of Information*

The efficient level of precision is that which maximizes total surplus at the time the object is sold. In our setup, total surplus is defined as the sum of the

[11]Bidders' private signals can be interpreted to be a reduced form of the following process. The seller provides public information about the characteristics of the object to be sold, and buyers combine it with their preferences to refine their estimates of their private valuations. These updated valuations are private information to buyers, as buyers are the only ones who know how the characteristics of the object that have been announced match their private preferences. Such a model is explored in Ganuza (2004).

[12]Our focus is on situations where the signals are obtained from public information or where the auctioneer must provide information symmetrically due to technological or regulatory constraints. Bergemann and Pesendorfer (2007) allowed the auctioneer to costlessly select each bidder's information structure, and showed that whenever possible some discrimination between bidders is optimal.

[13]The format of the auction is chosen without loss of generality, as the conditions for the revenue equivalence theorem are satisfied. However, the optimal mechanism could involve the use of a reserve price. Ganuza and Penalva (2004) used the linear experiment to study this model with a reserve price. The possibility of using a reserve price gives the auctioneer an additional tool to control bidders' informational rents. This raises his incentives to provide information and increases the optimal supply of information.

auctioneer's revenue and the expected utility of the bidder with the highest expected valuation at the time of the auction. As the price paid for the object is a pure transfer from the auctioneer to the winning bidder, total surplus is the expected valuation of the object by the winning bidder minus the cost of providing information. We first focus on the expected valuation of the winning bidder.

Denote the highest realization of the signal by $\pi_{1:n}$. The winner of the auction will be the bidder receiving $\pi_{1:n}$, so that his expected valuation is $V_1(n, \delta) = E[W_\delta(\Pi_{1:n})]$. Let $U_{1:n}(p)$ be the cumulative distribution function of the first-order statistic of n independent uniform random variables on $[0, 1]$, so that $U_{1:n}(p)$ is the cumulative distribution function of $\Pi_{1:n}$ and

$$V_1(n, \delta) = \int_0^1 W_\delta(p)\, dU_{1:n}(p).$$

As the auctioneer increases the precision of the signal, this expectation increases:

THEOREM 3: *For signals ordered in terms of integral precision, the expected valuation of the winning bidder is nondecreasing in the precision of the signal, δ.*

PROOF: Let $\psi(\pi) = W_\delta(\pi) - W_{\delta'}(\pi)$. We wish to show that

$$V_1(n, \delta) \geq V_1(n, \delta') \quad \Leftrightarrow \quad \int_0^1 \psi(\pi)\, dU_{1:n}(\pi) \geq 0.$$

Let $\Psi(\pi) = \int_0^\pi \psi(p)\, dp$. Integral precision implies $\Psi(\pi) \leq 0$ for all π. The result follows from $U_{1:n}(\pi) = \pi^n$, $\Psi(1) = 0$, and integration by parts:

$$\begin{aligned}\int_0^1 \psi(\pi)\, dU_{1:n}(\pi) &= \int_0^1 \psi(\pi) n\pi^{n-1}\, d\pi \\ &= -\int_0^1 \Psi(\pi) n(n-1)\pi^{n-2}\, d\pi \geq 0.\end{aligned}$$

Q.E.D.

Intuitively, more information increases the probability that the good is assigned to a bidder who, ex post, will value it more highly. Hence, a more precise signal leads to a more efficient allocation.

Theorem 3 implies that if the provision of information is costless, it is efficient to release all available information. With costly information, the trade-off faced when choosing the efficient level of precision, δ^E, is between increasing the efficiency of the allocation and the costs of providing information,

$$\delta_n^E = \arg\max_\delta V_1(n, \delta) - \delta.$$

This trade-off depends on the level of competition.

THEOREM 4: *For signals ordered in terms of supermodular precision, total surplus is supermodular in the precision of the signal, δ, and the number of bidders, n.*

PROOF: We wish to show that if $\delta > \delta'$, then $V_1(n, \delta) - V_1(n, \delta') = \int_0^1 \psi(\pi)\, dU_{1:n}(\pi) = E[\psi(\Pi_{1:n})]$ is nondecreasing in n. As greater supermodular precision implies $\psi(\pi)$ is nondecreasing in π and $\Pi_{1:n+1}$ stochastically dominates $\Pi_{1:n}$. Then $E[\psi(\Pi_{1:n+1})] \geq E[\psi(\Pi_{1:n})]$. *Q.E.D.*

Theorem 4 states that the improvement in total surplus from greater precision is nondecreasing in the number of bidders.[14] Thus, more bidders increase the social value of information. The intuition is that having more draws from the pool of bidder preferences increases the expected value of the winning bidder. This increases the social incentives to avoid misassigning the object by providing more information. Consequently, with fiercer competition it is efficient to spend more on the provision of information, that is, δ_n^E is nondecreasing in n. As δ_n^E may not be a singleton, we use Veinott's strong set order ($\delta_{n+1}^E \geq \delta_n^E$ if and only if $\forall \delta \in \delta_{n+1}^E, \delta' \in \delta_n^E, \max\{\delta, \delta'\} \in \delta_{n+1}^E$, and $\min\{\delta, \delta'\} \in \delta_n^E$) to make this statement precise.

COROLLARY 2: *For signals ordered in terms of supermodular precision, the efficient levels of precision, δ_n^E, are monotone nondecreasing in the number of bidders.*

This follows from Theorem 4 and Milgrom and Shannon (1994, Theorem 4).[15]

3.3. *The Auctioneer's Optimal Information Release*

The auctioneer chooses the level of precision to maximize his expected profits from the auction. Let $\pi_{2:n}$ denote the second-highest signal. The price in the auction is determined by the bidder receiving $\pi_{2:n}$. Thus, the expected price is $V_2(n, \delta) = E[W_\delta(\Pi_{2:n})]$. Let $U_{2:n}(p)$ be the cumulative distribution function of the second-order statistic of n independent uniform random variables on $[0, 1]$. $U_{2:n}(p)$ is the cumulative distribution function of $\Pi_{2:n}$. The expected price in the auction is

$$V_2(n, \delta) = \int_0^1 W_\delta(p)\, dU_{2:n}(p).$$

[14]We are assuming that the cost of providing information does not depend on the number of bidders. It is clear that if there is a cost of providing information to each extra bidder, an additional trade-off will arise.

[15]If we add the assumption that a change in δ leads to a nontrivial change in $W_\delta(\pi)$, then for $\delta > \delta'$, there exists $a, b \in (0, 1)$, $a < b$, such that $\forall p \in (a, b)$, $W_\delta(p) - W_{\delta'}(p) \neq 0$, it can be shown that $V_1(\delta, n)$ is strictly supermodular and every selection from δ^E is nondecreasing in n. The comparative statics results that will be established below will be strengthened in the same way. See the companion paper (Ganuza and Penalva (2009)).

The effect of increasing precision on the expected price depends on the level of competition.

THEOREM 5: *For signals ordered in terms of integral precision, the following holds*:

(i) *If* $n = 2$, *the expected price is nonincreasing in the precision of the signal*: $\delta > \delta'$ *implies* $V_2(2, \delta) \leq V_2(2, \delta')$.

(ii) *For all* $\delta' < \delta$, *there exists* n' *such that for all* $n > n'$, *the more precise signal produces a higher expected price*, $V_2(n, \delta) \geq V_2(n, \delta')$.

PROOF: (i) By the law of iterated expectations, the expected value of the distribution of expected valuations does not depend on δ. If $n = 2$, then

$$\mu = \frac{V_1(2, \delta) + V_2(2, \delta)}{2} = \frac{V_1(2, \delta') + V_2(2, \delta')}{2}$$
$$\Rightarrow \quad V_1(2, \delta) - V_1(2, \delta') = -(V_2(2, \delta) - V_2(2, \delta')).$$

From Theorem 3, $V_1(2, \delta) - V_1(2, \delta') \geq 0$, so that $V_2(2, \delta) - V_2(2, \delta') \leq 0$.

(ii) We wish to show that there exists n' such that for all $n > n'$, $V_2(n, \delta) \geq V_2(n, \delta')$.

If for all π, $\psi(\pi) = 0$ there is nothing to prove. Suppose that $\psi(\pi) \neq 0$ on a Lebesgue measurable subset of $[0, 1]$. Let $\phi(n) \equiv V_2(n, \delta) - V_2(n, \delta')$,

$$\phi(n) = n(n-1) \int_0^1 \psi(\pi)(1-\pi)\pi^{n-2}\, d\pi.$$

Because Π_δ is more integral precise than $\Pi_{\delta'}$, then $\forall p \in (0, 1)$, $\Psi(p) \leq 0$. As $\Psi(1) = 0$, $\forall p \in [0, 1]$, $\int_p^1 \psi(\pi)\, d\pi \geq 0$. Let $A^- = \{\pi \in [0, 1] | \psi(\pi) < 0\}$ and $\hat{\pi} = \sup(A^-)$. Integral precision and the nontriviality condition imply that $\hat{\pi} \in (0, 1)$ and there exist $p_1, p_2 \in (\hat{\pi}, 1]$ such that $\forall \pi \in [p_1, p_2]$, $\psi(\pi) > 0$. Let $c_1 = \min_{\pi \in [0, p_1)} \psi(\pi)(1-\pi)$ and $c_2 = \min_{\pi \in [p_1, p_2]} \psi(\pi)(1-\pi)$. Notice that $c_1 < 0$ and $c_2 > 0$. Then

$$\begin{aligned}
\phi(n) &= n(n-1) \\
&\quad \times \left(\int_0^{p_1} \psi(\pi)(1-\pi)\pi^{n-2}\, d\pi + \int_{p_1}^1 \psi(\pi)(1-\pi)\pi^{n-2}\, d\pi \right) \\
&\geq n(n-1)\left(\int_0^{p_1} c_1 \pi^{n-2}\, d\pi + \int_{p_1}^1 c_2 \pi^{n-2}\, d\pi \right) \\
&\geq n(n-1)\left(\int_0^{p_1} c_1 \pi^{n-2}\, d\pi + \int_{p_1}^{p_2} c_2 \pi^{n-2}\, d\pi \right) \\
&= n[p_1^{n-1} c_1 + (p_2^{n-1} - p_1^{n-1}) c_2] \\
&= n p_2^{n-1}[(p_1/p_2)^{n-1}(c_1 - c_2) + c_2].
\end{aligned}$$

Let

$$\hat{n} \equiv 1 + \frac{\ln\left(\dfrac{c_2}{c_2 - c_1}\right)}{\ln\left(\dfrac{p_1}{p_2}\right)}.$$

Since $p_1/p_2 < 1$, then for all $n > \hat{n}$, $(p_1/p_2)^{n-1}(c_1 - c_2) + c_2 > 0$ and $\phi(n) > 0$. *Q.E.D.*

Notice that when the number of bidders is small, increasing the precision of the signal can reduce the expected price. Then, even if information is costless, the auctioneer prefers not to release any information. The intuition for this result is that increasing precision has two effects on the price: it increases the willingness to pay by the winning bidder, which increases the price, but it also increases informational rents, which lowers the price. Eventually, when the number of bidders is sufficiently high, the effect on efficiency overwhelms the effect on informational rents and information becomes valuable to the auctioneer.[16]

To formalize the effect of a higher δ on informational rents, let $R_w(n, \delta)$ denote the expected informational rents of the winning bidder. $R_w(n, \delta)$ is equal to $V_1(n, \delta) - V_2(n, \delta)$, the difference between the expected valuation of the winning bidder and that of the bidder with the second-highest realization of the private signal.

PROPOSITION 6: *For signals ordered in terms of supermodular precision, the expected informational rents of the winning bidder are nondecreasing in the precision of the signal*, δ.

PROOF[17]: We wish to show that for $\delta > \delta'$, $V_1(n, \delta) - V_2(n, \delta) \geq V_1(n, \delta') - V_2(n, \delta')$, which is equivalent to showing $E[\psi(\Pi_{1:n})] \geq E[\psi(\Pi_{2:n})]$. This inequality follows from the fact that $\psi(\pi)$ is nondecreasing in π and $\Pi_{1:n}$ stochastically dominates $\Pi_{2:n}$. *Q.E.D.*

[16]A related result can be found in Board (2009). He showed that the auctioneer, when deciding whether or not to provide an additional piece of information, will always choose not to reveal it if there are only two bidders. He also showed that as the number of bidders goes to infinity, the information will be revealed.

[17]Proposition 6 can also be proven using the fact that $X \geq_{\text{disp}} Y$ implies the spacings of X (the difference between order statistics) stochastically dominate those of Y (Theorem 3.B.31 in Shaked and Shantikumar (2007)). Our proofs do not make explicit use of this property of the dispersive order. We rely on the additional fact that we are comparing random variables with the same mean, $E[X] = E[Y]$. This has allowed us to obtain results on the effect of increasing dispersion on the expected values of order statistics that, as far as we know, have not been explored in the statistics literature.

Greater precision makes the distribution of expected valuations more disperse, that is, it makes bidders more heterogeneous, which translates into higher informational rents for the winning bidder.

The auctioneer's problem is to choose the level of precision, δ^A, which maximizes his expected profits, that is, the difference between the expected price and the cost of providing more information:

$$\delta_n^A = \arg\max_\delta V_2(n, \delta) - \delta.$$

We find that, as with total surplus, the auctioneer's profits exhibit a complementarity between the level of precision and the number of bidders.

THEOREM 6: *For signals ordered in terms of supermodular precision, the auctioneer's expected profits are supermodular in the precision of the signal, δ, and the number of bidders, n.*

PROOF: We wish to show that for $\delta > \delta'$, $V_2(n+1, \delta) - V_2(n+1, \delta') \geq V_2(n, \delta) - V_2(n, \delta')$, which is equivalent to showing $E[\psi(\Pi_{2:n+1})] \geq E[\psi(\Pi_{2:n})]$. This inequality follows from the fact that $\psi(\pi)$ is nondecreasing in π and $\Pi_{2:n+1}$ stochastically dominates $\Pi_{2:n}$. *Q.E.D.*

COROLLARY 3: *For signals ordered in terms of supermodular precision, the optimal levels of precision, δ_n^A, are monotone nondecreasing in the number of bidders.*

The proof is immediate from Theorem 6 above and Milgrom and Shannon (1994, Theorem 4).

3.4. *Optimal versus Efficient Provision of Information*

By comparing private incentives to provide information with social ones, we obtain the following theorem.

THEOREM 7: *For signals ordered in terms of supermodular precision, the optimal levels of precision are lower than the efficient levels: $\delta_n^A \leq \delta_n^E$. The difference between the efficient and the optimal levels disappears as the number of bidders goes to infinity.*

PROOF: Rewrite the auctioneer's problem as

$$\delta^A = \arg\max_\delta V_1(n, \delta) - \delta - R_w(n, \delta).$$

Consider $\delta \in \delta_n^E$ and $\delta'_n \in \delta^A$. We prove $\delta_n^E \geq \delta_n^A$ by contradiction: $\delta^E \geq \delta^A$ implies $\max\{\delta, \delta'\} \in \delta^E$ and $\min\{\delta, \delta'\} \in \delta^A$. Suppose this is not true. In particular, suppose $\delta < \delta'$, but $\max\{\delta, \delta'\} = \delta' \notin \delta^E$. Then by the optimality of

δ', $V_1(n, \delta') - \delta' - R_w(n, \delta') \geq V_1(n, \delta) - \delta - R_w(n, \delta)$. As informational rents, $R_w(n, \delta)$, are nondecreasing in δ (Proposition 6), $\delta < \delta'$ implies $V_1(n, \delta') - \delta' \geq V_1(n, \delta) - \delta$, which contradicts the initial hypothesis: $\delta' \notin \delta^E$. Furthermore, $\delta < \delta'$ and $\min\{\delta, \delta'\} = \delta \notin \delta^A$ also lead to a contradiction by a similar argument.

The second part follows from the fact that informational rents disappear as n goes to infinity:

$$R_w(n, \delta) = V_1(n, \delta) - V_2(n, \delta) = \int_0^1 W_\delta(\pi)\, d(U_{1:n}(\pi) - U_{2:n}(\pi)).$$

As $W_\delta(\pi)$ is bounded and $U_{1:n}(\pi) - U_{2:n}(\pi) = n(\pi^n - \pi^{n-1}) \Rightarrow \lim_{n\to\infty} U_{1:n}(\pi) - U_{2:n}(\pi) = 0$, then $R_w(n, \delta)$ converges to zero. *Q.E.D.*

This formulation clarifies the trade-off faced by the auctioneer when providing information to the market. On the one hand, when the auctioneer increases precision, the efficiency of the allocation rises ($V_1(n, \delta)$ is nondecreasing in δ (Theorem 3)). On the other hand, the increase in precision also raises the informational rents of the winning bidder ($R_w(n, \delta)$ is nondecreasing in δ (Proposition 6)). The optimal balance of these two opposing effects leads the auctioneer to provide lower precision than would be efficient. In other words, the auctioneer will restrict the supply of information to the market so as to make bidders more homogeneous, with the underlying goal of intensifying competition and increasing his expected revenue.

The auctioneer's trade-off is also affected by the number of bidders. More bidders increase the positive effect of precision on expected revenues—a more efficient allocation—and reduce the negative effect—informational rents. The compounded effect is to increase the incentives of the auctioneer to reveal information so that as the number of bidders increases, so does the optimal amount of precision. In the limit, as the number of bidders goes to infinity, informational rents disappear and with them, the difference between the efficient and the optimal level of precision also disappears.

3.5. *Integral Precision and Costless Provision of Information*

If a higher δ implies only that the signal is more integral precise, then the supermodularity of $V_1(n, \delta)$ and $V_2(n, \delta)$ is lost and the model is silent on the comparative statics on the incentives of the auctioneer with costly provision of information. However, we have shown that Theorem 3 and Theorem 5 hold with integral precision. These results imply that with costless provision of information, the incentives of the auctioneer to supply information will be weakly lower than the efficient ones. On the one hand, it is efficient to supply all available information (Theorem 3), while on the other hand, information may have a negative value for the auctioneer if the number of bidders is small (Theorem 5). Furthermore, from the second part of Theorem 5, it follows that both the efficient and the optimal level of information converge as the number of bidders goes to infinity.

4. FINAL REMARKS AND ALTERNATIVE SIGNAL ORDERINGS

This paper presents a new approach to ranking signals based on the idea that the informational content of a signal is reflected in the dispersion of the distribution of posterior conditional expectations. Using two nested variability orders (the dispersive and convex orders), two ordered informativeness criteria are characterized (supermodular precision $\succ$ integral precision). An important property of integral precision is that it is implied by all standard orders based on the value of information. Supermodular precision provides a powerful tool for comparative statics and can be applied in commonly used information models with linear conditional expectations, such as the normal and linear experiments.

Other signal orderings can be constructed using alternative notions of variability. The following three notions of variability are grouped by the idea of single-crossing distribution functions and have been used to study information problems in several settings.

DEFINITION 3—Single-Crossing Variability Criteria: Let Y and Z be two real-valued random variables with distributions F and G, respectively.

- *sMPS* (Diamond and Stiglitz (1974)): Y is a simple mean-preserving spread (sMPS) of Z ($Y \geq_{\text{sMPS}} Z$) if for some $v^\dagger$, $v > (<)\, v^\dagger \Rightarrow G(v) - F(v) \geq (\leq)\, 0$.
- *Single-Crossing (SC) dispersion*: Y is more SC disperse than Z ($Y \geq_{\text{sc}} Z$) if for all $v > v'$, $G(v') - F(v') \geq (>)\, 0$ implies $G(v) - F(v) \geq (>)\, 0$.
- *Rotation* (Johnson and Myatt (2006)): Y is obtained from Z by a rotation ($Y \geq_{\text{rot}} Z$) if for all $v > v'$, $G(v') - F(v') \geq 0$ implies $G(v) - F(v) > 0$.[18]

These notions of variability differ only in the conditions they place on the set of points where the distribution functions of Y and Z can touch, $\mathcal{A}_0 = \{v | F(v) - G(v) = 0\}$. A rotation allows $\mathcal{A}_0$ to contain at most one point, SC dispersion allows $\mathcal{A}_0$ to be an interval, and sMPS does not impose restrictions on $\mathcal{A}_0$. Hence, these criteria are nested. Furthermore, for random variables with the same means, sMPS implies the convex order, so that if $E[Y] = E[Z]$,

$$(1) \qquad Y \geq_{\text{rot}} Z \quad \Rightarrow \quad Y \geq_{\text{sc}} Z \quad \Rightarrow \quad Y \geq_{\text{sMPS}} Z \quad \Rightarrow \quad Y \geq_{\text{cx}} Z.$$

[18] Here is Johnson and Myatt's (2006) definition: For a family of distributions indexed by k, a local change in k leads to a rotation in G_k if for some $v_k^\dagger$, and each v, $v \lesseqgtr v_k^\dagger \iff \partial G_k(v)/\partial k \gtreqless 0$. In the context of pairwise comparisons of random variables, Johnson and Myatt's (2006) original definition corresponds to the strict single-crossing condition.

From the definitions, it is clear that none of them implies the dispersive order and that $Y \geq_{\text{disp}} Z \Rightarrow Y \geq_{\text{sc}} Z$. But the dispersive order allows $\mathcal{A}_0$ to be an interval and, hence, does not imply a rotation. Signal orderings based on these dispersion measures will preserve the implications in Equation (1).[19]

Like integral and supermodular precision, these additional criteria (applied as information orders) are useful to study situations in which a principal (seller) controls the information available to agents (buyers) and the agent's action is a function of his conditional expectation. SC dispersion is used in the companion paper (Ganuza and Penalva (2009)) to provide an additional precision criterion—single-crossing precision. Single-crossing precision adds to the results obtained with integral precision that the optimal amount of information is weakly monotonic in the number of bidders. Johnson and Myatt (2006) introduced the notion of a rotation in the demand curve. They showed that the monopolist's information (advertising) policy is one of the forces that generate a rotation in the demand curve and they used rotations to explain the results in Lewis and Sappington (1994). Szalay (2009) used sMPS to study how a principal's choice of contracts affects the agent's incentives to acquire information.

APPENDIX

PROOF OF THEOREM 1: (i) MIO-ND implies

$$\int_0^\pi \big(F_1(v|F_1^{-1}(p)) - F_2(v|F_2^{-1}(p))\big)\,dp \geq 0 \quad \forall \pi \in [0,1].$$

Using the properties of Riemann–Stieltjes integrals, integrating by parts, and exchanging integrations limits yields

$$\begin{aligned}&\int_0^\pi (W_1(p) - W_2(p))\,dp \\ &\quad = -\int_0^\pi \left(\int_{\mathcal{V}} \big(F_1(v|F_1^{-1}(p)) - F_2(v|F_2^{-1}(p))\big)\,dv\right) dp \\ &\quad = -\int_{\mathcal{V}} \int_0^\pi \big(\big(F_1(v|F_1^{-1}(p)) - F_2(v|F_2^{-1}(p))\big)\,dp\big)\,dv.\end{aligned}$$

Thus, MIO-ND implies the integrand is positive and the result follows.[20]

[19]Regarding these signal orderings, it is important to note that (i) comparisons in terms of single-crossing are not transitive (see Chateauneuf, Cohen, and Meilijson (2004)) and (ii) there are monotone signals that are Blackwell ordered but whose distributions of posterior conditional expectations cross several times (see the proof of Theorem 1(ii)).

[20]This proof has been included for completeness. A similar proof can be found in Shaked and Shantikumar (2007) in the context of the positive quadrant dependence order.

(ii) Signals generated by monotone partitions can be Blackwell ordered and not be more supermodular precise. A numerical example of this is the following: Let V be uniformly distributed on [0, 1]. Let X be equal to 0 if $v \in [0, 1/2)$ and equal to 1 if $v \in [1/2, 1]$. Then $E[v|X = 0] = 1/4$ and $E[v|X = 1] = 3/4$. Similarly, let Y be distributed as

$$Y = \begin{cases} 0, & \text{if } v \in [0, 1/4), \\ 1, & \text{if } v \in [1/4, 1/2), \\ 2, & \text{if } v \in [1/2, 3/4), \\ 3, & \text{if } v \in [3/4, 1], \end{cases} \quad \Rightarrow \quad E[v|Y] = \begin{cases} 1/8, & \text{if } Y = 0, \\ 3/8, & \text{if } Y = 1, \\ 5/8, & \text{if } Y = 2, \\ 7/8, & \text{if } Y = 3. \end{cases}$$

Y is based on a finer partition than X, so it is sufficient for X, but the conditional expectation functions will be

$$W_X(\pi) = \begin{cases} 1/4, & \text{if } \pi \in [0, 1/2), \\ 3/4, & \text{if } \pi \in [1/2, 1], \end{cases}$$

$$W_Y(\pi) = \begin{cases} 1/8, & \text{if } \pi \in [0, 1/4), \\ 3/8, & \text{if } \pi \in [1/4, 1/2), \\ 5/8, & \text{if } \pi \in [1/2, 3/4), \\ 7/8, & \text{if } \pi \in [3/4, 1]. \end{cases}$$

Let $\psi(\pi) \equiv W_X(\pi) - W_Y(\pi)$. Then

$$\psi(\pi) = \begin{cases} -1/8, & \text{if } \pi \in [0, 1/4), \\ 1/8, & \text{if } \pi \in [1/4, 1/2), \\ -1/8, & \text{if } \pi \in [1/2, 3/4), \\ 1/8, & \text{if } \pi \in [3/4, 1], \end{cases}$$

and crosses three times, so that X and Y are not ordered in terms of supermodular precision. *Q.E.D.*

PROOF OF PROPOSITION 2: Consider the two signals X_1 and X_2 as defined in Table A.1. Direct computation shows that they satisfy the monotone likelihood ratio property.

Given a uniform prior, X_1 and X_2 are also uniformly distributed, that is, $\Pr(X_k = x) = 1/3$ for $x = 0, 1, 2$. Note that $F_1(X_1) = F_2(X_2)$. $F_1(x, v)$ is more

TABLE A.1
LIKELIHOOD FUNCTIONS FOR X_1 AND X_2 (NUMBERS IN 1/81THS)

	$[X_1 \| v]$	1	2	3	4
X_1	0	48	30	24	6
	1	27	27	27	27
	2	6	24	30	48

	$[X_2 \| v]$	1	2	3	4
X_2	0	42	40	16	10
	1	27	27	27	27
	2	12	14	38	44

TABLE A.2
CONDITIONAL DISTRIBUTIONS $F_k(V|X_k \leq x)$

		1	2	3	4
X_1	0	0.44	0.72	0.94	1
	1	0.35	0.61	0.85	1
	2	0.25	0.50	0.75	1

		1	2	3	4
X_2	0	0.39	0.76	0.91	1
	1	0.32	0.63	0.83	1
	2	0.25	0.50	0.75	1

informative than $F_2(x, v)$ in terms of MIO-ND if and only if $F_1(v|X_1 \leq i) \geq_{\text{st}} F_2(v|X_2 \leq i)$ for $i = 0, 1, 2$. This condition does not hold, as can be verified by inspecting the conditional cumulative distributions in Table A.2.

Nevertheless, X_1 is more supermodular precise than X_2 with respect to the uniform prior, as can be seen by computation of conditional expectations:

$$E[V|X_1] = (\,34 \quad 45 \quad 56\,)/18 \quad \text{and}$$
$$E[V|X_2] = (\,35 \quad 45 \quad 55\,)/18. \qquad \textit{Q.E.D.}$$

PROOF OF PROPOSITION 3: Using Bayes' rule and given that Π_k is uniformly distributed,

$$W_k(\pi) = E[V|\Pi_k = \pi] = v_L + (v_H - v_L)\hat{f}_k(\pi|H)q. \tag{2}$$

(i) For integral precision, we integrate over π in Equation (2):

$$\int^{\pi} W_k(p)\,dp = v_L + (v_H - v_L)\int^{\pi} \hat{f}_k(p|H)q\,dp$$
$$= v_L + q(v_H - v_L)\hat{F}_k(\pi|H),$$
$$\int^{\pi} W_1(p)\,dp \leq \int^{\pi} W_2(p)\,dp \quad \iff \quad \hat{F}_1(\pi|H) \leq \hat{F}_2(\pi|H).$$

(ii) For supermodular precision, the result is immediate from Equation (2):

$$W_1(\pi) - W_2(\pi) \text{ is nondecreasing in } \pi$$
$$\iff \quad \hat{f}_1(\pi|H) - \hat{f}_2(\pi|H) \text{ is nondecreasing in } \pi. \qquad \textit{Q.E.D.}$$

PROOF OF THEOREM 2: ($\Leftarrow$) This follows immediately from Theorem 1(i).

($\Rightarrow$) We use the characterization of sufficiency for dichotomous random variables in terms of the power of the most powerful test of size α (see Jewitt (2007)): X_1 is sufficient for X_2 if and only if for every $\alpha \in (0, 1)$,

$$\beta_1(\alpha) = 1 - F_1(F_1^{-1}(\alpha|v_L)|v_H) \geq 1 - F_2(F_2^{-1}(\alpha|v_L)|v_H) = \beta_2(\alpha).$$

As Π_k is uniform on [0, 1],

$$
\begin{aligned}
(3)\qquad & q\hat{f}_k(\pi|H) + (1-q)\hat{f}_k(\pi|L) = 1 \\
& \iff \quad \hat{f}_k(\pi|H) = \frac{1}{q} - \frac{1-q}{q}\hat{f}_k(\pi|L), \\
& \Rightarrow \quad \hat{F}_k(\pi|H) = \frac{1}{q}\pi - \frac{1-q}{q}\hat{F}_k(\pi|L) \quad \text{and} \\
& \qquad \hat{F}_k(\pi|L) = \frac{1}{1-q}\pi - \frac{q}{1-q}\hat{F}_k(\pi|H).
\end{aligned}
$$

Set $\pi = \hat{F}_1^{-1}(\alpha|L)$ for $k = 1, 2$. Then

$$
(4)\qquad \alpha = \hat{F}_1^{-1}(\alpha|L)\frac{1}{1-q} - \frac{q}{1-q}\hat{F}_1(\hat{F}_1^{-1}(\alpha|L)|H),
$$

$$
(5)\qquad \alpha = \hat{F}_2^{-1}(\alpha|L)\frac{1}{1-q} - \frac{q}{1-q}\hat{F}_2(\hat{F}_2^{-1}(\alpha|L)|H).
$$

Integral precision is characterized by $\forall\pi$, $\hat{F}_1(\pi|H) \leq \hat{F}_2(\pi|H)$, which from Equation (3) is equivalent to $\forall\pi$, $\hat{F}_1(\pi|L) \geq \hat{F}_2(\pi|L)$, that is, $\forall\pi$, $\hat{F}_1^{-1}(\pi|L) \leq \hat{F}_2^{-1}(\pi|L)$. By equations (4) and (5), $\hat{F}_1^{-1}(\pi|L) \leq \hat{F}_2^{-1}(\pi|L)$ implies $\hat{F}_1(\hat{F}_1^{-1}(\alpha|L)|H) \leq \hat{F}_2(\hat{F}_2^{-1}(\alpha|L)|H)$. Then greater integral precision implies that Π_1 has greater power and, hence, is more Blackwell informative than Π_2. As Π_1 is sufficient for Π_2 if and only if X_1 is sufficient for X_2, then the result follows. *Q.E.D.*

PROOF OF PROPOSITION 4: The result follows from the fact that the dispersive order is location-free and that for any random variable X and $a \geq 1$, $aX \geq_{\text{disp}} X$ (Shaked and Shantikumar (2007, Theorem 3.B.4)). *Q.E.D.*

PROOF OF PROPOSITION 5: We use the following fact about differentiable copulas (where Π_k is uniform on [0, 1]): $F_k(v|\Pi_k = \pi) = \partial C_k(H(v), \pi)/\partial\pi$. If $\forall u$, $-\partial C_k(u, \pi)/\partial\pi$ is supermodular in (π, k), then for all $H(v)$ and for all v, $-F_k(v|\Pi_k = \pi)$ is supermodular in (k, π). Consider applying integration by parts on $W_k(\pi) = E[V|\Pi_k = \pi]$:

$$
W_k(\pi) = \int_v v\,dF_k(v|\Pi_k = \pi) = 1 - \int_v F_k(v|\Pi_k = \pi)\,dv.
$$

Then, by the preservation of supermodularity by integration, $W_k(\pi)$ is supermodular. *Q.E.D.*

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